Mathematics and Sudoku V

Dedicated to the memory of Professor Sibe Mardešić

KITAMOTO Takuya, WATANABE Tadashi

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We discuss on the worldwide famous Sudoku by using mathematical approach. This paper is the 5th paper in our series, so we use the same notations and terminologies in [1] –[4] without any descriptions.

9. Classification of intersectable systems II.

This section is a continuous section of previous section 8. We consider some basic relations among some types in the section 8.

Proposition 45. (Type 3) For each intersectable system $\omega=(S,T)$ of Type 3, we have an intersectable system $\omega_1=(S_1,T_1)$ of Type 2 such that $T_\omega=T_{\omega_1}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in \boldsymbol{SOL}(f_0)$.

Proof. We take an intersectable system (S,T) of Type 3. First, we assume that $S=\{s_1,s_2\}$ and $T=\{t_1,t_2\}$ such that

(1) s_1 and s_2 are 3×3 blocks,

(2) t_1 and t_2 are columns,

(3) $s_1\cap s_2=\phi$,

(4) $t_1\cap t_2=\phi$,

(5) $s_i\cap t_j\neq\phi$ for each i,j, $1\leqq i\leqq 2, 1\leqq j\leqq 2$.

By the above conditions (1)–(5) there exist the unique 3×3 block u and the unique column v such that

(6) $s_1\cap u=\phi$ and $s_2\cap u=\phi$,

(7) $t_1\cap v=\phi$ and $t_2\cap v=\phi$,

(8) $s_1\cup s_2\cup u=t_1\cup t_2\cup v$.

We put $S_1=\{u\}$ and $T_1=\{v\}$. Since $u\cap v\neq\phi$ by (8), $\omega_1=(S_1,T_1)$ is an intersectable system of Type 2. Take any sudoku matrix $K=(K_\alpha)_{\alpha\in J_1\times J_2}\in SMTX(f,f_0)$ and we put $K'=T_\omega(K)=T(S,T)(K)$ and $K''=T_{\omega_1}(K)=T(S_1,T_1)(K)$. Thus by definitions, we have

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$$(9)\ K'_\alpha=\begin{cases}K_\alpha & for\ \alpha\in(J_1\times J_2-(s_1\cup s_2)\cup(t_1\cup t_2))\cup((s_1\cup s_2)\cap(t_1\cup t_2))\\ K_\alpha\cap K_{t_1\cup t_2-s_1\cup s_2} & for\ \alpha\in s_1\cup s_2-t_1\cup t_2\\ K_\alpha\cap K_{s_1\cup s_2-t_1\cup t_2} & for\ \alpha\in t_1\cup t_2-s_1\cup s_2\end{cases},$$

$$(10)\ K''_\alpha=\begin{cases}K_\alpha & for\ \alpha\in(J_1\times J_2-u\cup v)\cup(u\cap v)\\ K_\alpha\cap K_{v-u} & for\ \alpha\in u-v\\ K_\alpha\cap K_{u-v} & for\ \alpha\in v-u\end{cases}.$$

By (1)−(8) we have

(11) $J_1\times J_2-(s_1\cup s_2)\cup(t_1\cup t_2)=(J_1\times J_2-s_1\cup s_2\cup u)\cup(s_1\cup s_2\cup u-(s_1\cup s_2)\cup(t_1\cup t_2))$,

(12) $s_1\cup s_2\cup u-(s_1\cup s_2)\cup(t_1\cup t_2)=u-t_1\cup t_2=u\cap v$,

(13) $(s_1\cup s_2)\cap(t_1\cup t_2)=s_1\cup s_2\cup u-u\cup v$.

Thus, by (11)−(13) we have

$(J_1\times J_2-(s_1\cup s_2)\cup(t_1\cup t_2))\cup((s_1\cup s_2)\cap(t_1\cup t_2))$

$=(J_1\times J_2-s_1\cup s_2\cup u)\cup(u\cap v)\cup(s_1\cup s_2\cup u-u\cup v)$

$=(J_1\times J_2-u\cup v)\cup(u\cap v)$, that is,

(14) $(J_1\times J_2-(s_1\cup s_2)\cup(t_1\cup t_2))\cup((s_1\cup s_2)\cap(t_1\cup t_2))=(J_1\times J_2-u\cup v)\cup(u\cap v)$.

Moreover, by (1)−(8) we have

(15) $s_1\cup s_2-t_1\cup t_2=v-u$,

(16) $t_1\cup t_2-s_1\cup s_2=u-v$.

By (9), (10), (14), (15), (16) we have

(17) $K'_\alpha=K''_\alpha$ for each $\alpha\in J_1\times J_2$.

By (17) we have that $T_\omega(K)=T_{\omega_1}(K)$ for each $K=(K_\alpha)_{\alpha\in J_1\times J_2}\in SMTX(f,f_0)$, i.e.,

(18) $T_\omega=T_{\omega_1}$.

Thus, in this case we show the required one.

Secondly, we must consider the case,

(19) t and t_2 are rows.

In this case (19), by the similar way as the case (2) we can show (18). Hence we complete the proof of Proposition 45.

Proposition 46. (Type 4) For each intersectable system $\omega=(S,T)$ of Type 4 there exist intersectable systems $\omega_1=(S_1,T_1)$ and $\omega_2=(S_2,T_2)$ of Type 2 such that $T_\omega\geqq T_{\omega_2}\circ T_{\omega_1}\cap T_{\omega_1}\circ T_{\omega_2}$, i.e., $T(S,T)\geqq T(S_2,T_2)\circ T(S_1,T_1)\cap T(S_1,T_1)\circ T(S_2,T_2)$ in $STRF(f,f_0)$ for each $f\in SOL(f_0)$.

Proof. We take an intersectable system (S,T) of Type 4. First, we assume that $S=\{s_1,s_2\}$ and $T=\{t_1,t_2\}$ such that

(1) s_1 is a 3×3 block and s_2 is a row,

(2) t_1 and t_2 are columns,

(3) $s_1 \cap s_2 = \phi$,

(4) $t_1 \cap t_2 = \phi$,

(5) $s_i \cap t_j \neq \phi$ for each i, j, $1 \leqq i \leqq 2$, $1 \leqq j \leqq 2$.

By (1)–(5) we can choose sets $A_1 = \{i_1, i_2, i_3\} \subset J_1$, $A_2 = \{i_4, i_5, i_6\} \subset J_1$, $A_3 = \{i_7, i_8, i_9\} \subset J_1$ and sets $B_1 = \{j_1, j_2, j_3\} \subset J_2$, $B_2 = \{j_4, j_5, j_6\} \subset J_2$, $B_3 = \{j_7, j_8, j_9\} \subset J_2$ such that

(6) $\{A_1, A_2, A_3\} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}\}$,

(7) $\{B_1, B_2, B_3\} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}\}$,

(8) $s_1 = A_1 \times B_1$,

(9) $s_2 = \{i_4\} \times J_2$.

(10) $t_1 = J_1 \times \{j_1\}$,

(11) $t_2 = J_1 \times \{j_2\}$.

We put intersectable systems (S_1, T_1) and (S_2, T_2) of Type 2 as follows:

(12) $S_2 = \{s_2\}$ and $T_2 = \{v_1\}$, where $v_1 = A_2 \times B_1$,

(13) $S_1 = \{u\}$ and $T_1 = \{v_2\}$, where $u = J_1 \times \{j_3\}$ and $v_2 = A_3 \times B_1$.

Take any sudoku matrix $K = (K_\alpha)_{\alpha \in J_1 \times J_2} \in SMTX(f, f_0)$ and we put $K' = T(S,T)(K)$. Thus we have

(14) $s = s_1 \cup s_2$, $t = t_1 \cup t_2$,

(15) $$K'_\alpha = \begin{cases} K_\alpha & for\ \alpha \in (J_1 \times J_2 - s \cup t) \cup (s \cap t) \\ K_\alpha \cap K_{s-t} & for\ \alpha \in t - s \\ K_\alpha \cap K_{t-s} & for\ \alpha \in s - t \end{cases}.$$

We put $K^* = T(S_1, T_1)(K)$ and $K^{**} = T(S_2, T_2)(K^*)$. Thus we have

(16) $$K^*_\alpha = \begin{cases} K_\alpha & for\ \alpha \in (J_1 \times J_2 - u \cup v_2) \cup (u \cap v_2) \\ K_\alpha \cap K_{u-v_2} & for\ \alpha \in v_2 - u \\ K_\alpha \cap K_{v_2-u} & for\ \alpha \in u - v_2 \end{cases},$$

(17) $$K^{**}_\alpha = \begin{cases} K^*_\alpha & for\ \alpha \in (J_1 \times J_2 - s_2 \cup v_1) \cup (s_2 \cap v_1) \\ K^*_\alpha \cap K^*_{s_2-v_1} & for\ \alpha \in v_1 - s_2 \\ K^*_\alpha \cap K^*_{v_1-s_2} & for\ \alpha \in s_2 - v_1 \end{cases}.$$

We put $K^\# = T(S_2, T_2)(K)$ and $K^{\#\#} = T(S_1, T_1)(K^\#)$. Thus we have

(18) $$K^\#_\alpha = \begin{cases} K_\alpha & for\ \alpha \in (J_1 \times J_2 - s_2 \cup v_1) \cup (s_2 \cap v_1) \\ K_\alpha \cap K_{s_2-v_1} & for\ \alpha \in v_1 - s_2 \\ K_\alpha \cap K_{v_1-s_2} & for\ \alpha \in s_2 - v_1 \end{cases},$$

$$(19)\ K^{\#\#}_\alpha = \begin{cases} K^{\#}_\alpha & for\ \alpha\in(J_1\times J_2-u\cup v_2)\cup(u\cap v_2) \\ K^{\#}_\alpha\cap K^{\#}_{u-v_2} & for\ \alpha\in v_2-u \\ K^{\#}_\alpha\cap K^{\#}_{v_2-u} & for\ \alpha\in u-v_2 \end{cases}.$$

We put $K^{!}=(T(S_2,T_1)\circ T(S_1,T_1)\cap T(S_1,T_1)\circ T(S_2,T_2))(K)=K^{**}\cap K^{\#\#}$, i.e.,

(20) $K^{!}_\alpha=K^{**}_\alpha\cap K^{\#\#}_\alpha\ for\ \alpha\in J_1\times J_2$.

We must show

(21) $K^{!}\leqq K'$.

Claim 1. $K^{!}_\alpha\subset K'_\alpha$ for each $\alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)$.

Proof of Claim 1. Since $T(S_2,T_2)\circ T(S_1,T_1)\cap T(S_1,T_1)\circ T(S_2,T_2)$ is a sudoku transformation by Propositions 19, 20 and 42, we have

(22) $K^{!}\leqq K$.

Take any $\alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)$. By (22) we have

(23) $K^{!}_\alpha\subset K_\alpha$.

By (15) we have

(24) $K'_\alpha=K_\alpha$.

By (23) and (24) we have

(25) $K^{!}_\alpha\subset K'_\alpha$.

Hence, by (25), we have Claim 1.

Claim 2. $K^{!}_\alpha\subset K'_\alpha$ for each $\alpha\in s_2-v_1$.

Proof of Claim 2. By (1)–(13) we can easily show that

(25) $v_1-s_2=v_1\cap(t\cup u)-s_2=(v_1\cap t-s_2)\cup(v_1\cap u-s_2)$,

(26) $(v_1\cap t-s_2)\cup(v_2-u)=t-s$.

Since $(v_1\cap t-s_2)\cap(v_2\cup u)=\phi$, by (16) we have

(27) $K^{*}_\alpha=K_\alpha$ for each $\alpha\in v_1\cap t-s_2$.

Since $v_1\cap u-s_2\subset u-v_2$, by (16) we have

(28) $K^{*}_\alpha=K_\alpha\cap K_{v_2-u}$ for each $\alpha\in v_1\cap u-s_2$.

By (25),(26),(27),(28) we have

$K^{*}_{v_1-s_2}=K^{*}_{v_1\cap t-s_2}\cup K^{*}_{v_1\cap u-s_2}=K_{v_1\cap t-s_2}\cup(K_{v_1\cap u-s_2}\cap K_{v_2-u})\subset K_{v_1\cap t-s_2}\cup K_{v_2-u}=$
$K_{(v_1\cap t-s_2)\cup(v_2-u)}=K_{t-s}$.　i.e.,

(29) $K^{*}_{v_1-s_2}\subset K_{t-s}$.

Since $(s_2-v_1)\cap(v_2\cup u)=\phi$, by (16) we have

(30) $K^{*}_\alpha=K_\alpha$ for each $\alpha\in s_2-v_1$.

Take any $\alpha_0 \in s_2 - v_1$. By (17) we have

(31) $K^{**}_{\alpha_0} = K^{*}_{\alpha_0} \cap K^{*}_{v_1 - s_2}$.

By (29),(30),(31) we have that $K^{**}_{\alpha_0} = K^{*}_{\alpha_0} \cap K^{*}_{v_1 - s_2} = K_{\alpha_0} \cap K^{*}_{v_1 - s_2} \subset K_{\alpha_0} \cap K_{t-s}$, that is,

(32) $K^{**}_{\alpha_0} \subset K_{\alpha_0} \cap K_{t-s}$.

Since $\alpha_0 \in s_2 - v_1 \subset s - t$, by (15) we have

(33) $K'_{\alpha_0} = K_{\alpha_0} \cap K_{t-s}$.

By (32),(33) we have

(34) $K^{**}_{\alpha_0} \subset K'_{\alpha_0}$.

By (20),(34) we have that

(35) $K^{!}_{\alpha_0} = K^{**}_{\alpha_0} \cap K^{\#\#}_{\alpha_0} \subset K^{**}_{\alpha_0} \subset K'_{\alpha_0}$.

(35) means that Claim 2 holds.

Claim 3. $K^{!}_{\alpha} \subset K'_{\alpha}$ for each $\alpha \in (s_1 \cap u) \cup (s_2 \cap u)$.

Proof of Claim 3. Since $(s_1 \cap u) \cup (s_2 \cap u) \subset u - v_2$, by (16) we have

(36) $K^{*}_{\alpha} = K_{\alpha} \cap K_{v_2 - u}$ for each $\alpha \in (s_1 \cap u) \cup (s_2 \cap u)$.

Since $(s_1 \cap u) \cap (v_1 \cup s_2) = \phi$ and $s_2 \cap u \subset v_1 \cap s_2$, then we have

(37) $(s_1 \cap u) \cup (s_2 \cap u) \subset (J_1 \times J_2 - v_1 \cup s_2) \cup (v_1 \cap s_2)$.

By (37) and (17) we have

(38) $K^{**}_{\alpha} = K^{*}_{\alpha}$ for each $\alpha \in (s_1 \cap u) \cup (s_2 \cap u)$.

By (1)$-$(13) we have

(39) $v_2 - u \subset t - s$.

Take any $\alpha_0 \in (s_1 \cap u) \cup (s_2 \cap u)$. By (36),(38),(39) we have

(40) $K^{**}_{\alpha_0} = K^{*}_{\alpha_0} = K_{\alpha_0} \cap K_{v_2 - u} \subset K_{\alpha_0} \cap K_{t-s}$.

Since $\alpha_0 \in (s_1 \cap u) \cup (s_2 \cap u) \subset s - t$, by (15) we have

(41) $K'_{\alpha_0} = K_{\alpha_0} \cap K_{t-s}$.

By (40),(41) we have

(42) $K^{**}_{\alpha_0} \subset K'_{\alpha_0}$.

By (20),(42) we have

(43) $K^{!}_{\alpha_0} = K^{**}_{\alpha_0} \cap K^{\#\#}_{\alpha_0} \subset K^{**}_{\alpha_0} \subset K'_{\alpha_0}$.

(43) means that Claim 3 holds.

Claim 4. $K^{!}_{\alpha} \subset K'_{\alpha}$ for each $\alpha \in v_1 \cap t - s_2$.

Proof of Claim 4. Since $v_1 \cap t - s_2 \subset v_1 - s_2$, by (18) we have

(44) $K_\alpha^\# = K_\alpha \cap K_{s_2 - v_1}$ for each $\alpha \in v_1 \cap t - s_2$.

Since $(v_1 \cap t - s_2) \cap (v_2 \cup u) = \phi$, by (19) we have

(45) $K_\alpha^{\#\#} = K_\alpha^\#$ for each $\alpha \in v_1 \cap t - s_2$.

Since $s_2 - v_1 \subset s - t$, we have

(46) $K_{s_2 - v_1} \subset K_{s-t}$.

Take any $\alpha_0 \in v_1 \cap t - s_2$. By (44),(45),(46) we have

(47) $K_{\alpha_0}^{\#\#} = K_{\alpha_0}^\# = K_{\alpha_0} \cap K_{s_2 - v_1} \subset K_{\alpha_0} \cap K_{s-t}$.

Since $\alpha_0 \in v_1 \cap t - s_2 \subset t - s$, by (15) we have

(48) $K'_{\alpha_0} = K_{\alpha_0} \cap K_{s-t}$.

By (47),(48) we have

(49) $K_{\alpha_0}^{\#\#} \subset K'_{\alpha_0}$.

By (20),(49) we have

(50) $K_{\alpha_0}^! = K_{\alpha_0}^{\#\#} \cap K_{\alpha_0}^{**} \subset K_{\alpha_0}^{\#\#} \subset K'_{\alpha_0}$.

(50) means that Claim 4 holds.

Claim 5. $K_\alpha^! \subset K'_\alpha$ for each $\alpha \in v_2 \cap t$.

Proof of Claim 5. By (37),(18) we have

(51) $K_\alpha^\# = K_\alpha$ for each $\alpha \in (s_1 \cap u) \cup (s_2 \cap u)$.

Since $v_1 \cap u - s_2 \subset v_1 - s_2$, by (18) we have

(52) $K_\alpha^\# = K_\alpha \cap K_{s_2 - v_1}$ for each $\alpha \in v_1 \cap u - s_2$.

By (1)−(13) we can easily show that

(53) $u - v_2 = \{(s_1 \cap u) \cup (s_2 \cap u)\} \cup \{v_1 \cap u - s_2\}$,

(54) $s - t = (s_1 \cap u) \cup (s_2 \cap u) \cup (s_2 - v_1)$.

By (51),(52),(53),(54) we have that

$$K_{u-v_2}^\# = K_{(s_1 \cap u) \cup (s_2 \cap u)}^\# \cup K_{v_1 \cap u - s_2}^\# = K_{(s_1 \cap u) \cup (s_2 \cap u)} \cup (K_{v_1 \cap u - s_2} \cap K_{s_2 - v_1})$$
$$\subset K_{(s_1 \cap u) \cup (s_2 \cap u)} \cup K_{s_2 - v_1} = K_{(s_1 \cap u) \cup (s_2 \cap u) \cup (s_2 - v_1)} = K_{s-t} \quad \text{, that is,}$$

(55) $K_{u-v_2}^\# \subset K_{s-t}$.

Since $(v_2 \cap t) \cap (v_1 \cup s_2) = \phi$, by (18) we have

(56) $K_\alpha^\# = K_\alpha$ for each $\alpha \in v_2 \cap t$.

Since $v_2 \cap t = v_2 - u$, by (19) we have

(57) $K_\alpha^{\#\#} = K_\alpha^\# \cap K_{u-v_2}^\#$ for each $\alpha \in v_2 \cap t$.

Take any $\alpha_0 \in v_2 \cap t$. By (55),(56),(57) we have

(58) $K_{\alpha_0}^{\#\#} = K_{\alpha_0}^\# \cap K_{u-v_2}^\# = K_{\alpha_0} \cap K_{u-v_2}^\# \subset K_{\alpha_0} \cap K_{s-t}$.

Since $\alpha_0 \in v_2 \cap t \subset t-s$, by (15) we have

(59) $K'_{\alpha_0} = K_{\alpha_0} \cap K_{s-t}$.

By (58),(59) we have

(60) $K^{\#\#}_{\alpha_0} \subset K'_{\alpha_0}$.

By (20),(60) we have

(61) $K^{!}_{\alpha_0} = K^{**}_{\alpha_0} \cap K^{\#\#}_{\alpha_0} \subset K^{\#\#}_{\alpha_0} \subset K'_{\alpha_0}$.

(61) means that Claim 5 holds.

We have the decomposition of $J_1 \times J_2$ into five parts as follows:

$J_1 \times J_2 = \{(J_1 \times J_2 - s \cup t) \cup (s \cap t)\} \cup \{s_2 - v_1\} \cup \{(s_1 \cap u) \cup (s_2 \cap u)\} \cup \{v_1 \cap t - s_2\} \cup \{v_2 \cap t\}$

By the above decomposition and Claims 1−5, we have Proposition 46 in the case with (1) and (2).

Secondly for another case of Type 4, by the similar ways we can show Proposition 46. Hence we complete the proof of Proposition 46.

Proposition 47. (Type 6) For each intersectable system $\omega = (S,T)$ of Type 6, we have that $T_\omega = 1$ in $STRF(f,f_0)$ for each $f \in SOL(f_0)$.

Proof. First we put $S = \{s_1,s_2,s_3\}$ and $T = \{t_1,t_2,t_3\}$ such that

(1) s_1,s_2,s_3 are 3×3 blocks,

(2) t_1,t_2,t_3 are columns,

(3) $s_i \cap s_j = \phi$ for $i \neq j$, $1 \leqq i \leqq 3$, $1 \leqq j \leqq 3$,

(4) $t_i \cap t_j = \phi$ for $i \neq j$, $1 \leqq i \leqq 3$, $1 \leqq j \leqq 3$,

(5) $s_i \cap t_j \neq \phi$ for i,j, $1 \leqq i \leqq 3, 1 \leqq j \leqq 3$.

By (1)−(5) we can easily show that

(6) $s = s_1 \cup s_2 \cup s_3 = t_1 \cup t_2 \cup t_3 = t$.

Take any sudoku matrix $K = (K_\alpha)_{\alpha \in J_1 \times J_2} \in SMTX(f,f_0)$ and we put $K' = T_\omega(K)$ $= T(S,T)(K)$. By definition we have

$$(7)\ K'_\alpha = \begin{cases} K_\alpha & for\ \alpha \in (J_1 \times J_2 - s \cup t) \cup (s \cap t) \\ K_\alpha \cap K_{t-s} & for\ \alpha \in s-t \\ K_\alpha \cap K_{s-t} & for\ \alpha \in t-s \end{cases}.$$

By (6) we have

(8) $(J_1 \times J_2 - s \cup t) \cup (s \cap t) = (J_1 \times J_2 - s) \cup s = J_1 \times J_2$,

(9) $s - t = \phi,\ t - s = \phi$.

By (7),(8),(9) we have

(10) $K'_\alpha = K_\alpha$ for $\alpha \in J_1 \times J_2$, i.e.,

(11) $T_\omega = 1$.

Thus we have (11) for the case of conditions (1) and (2).

By the same way we can show (11) for the other case of Type 6. Hence we have Proposition 47.

Proposition 48. (Type 7) For each intersectable system $\omega=(S,T)$ of Type 7 there exists an intersectable systems $\omega_1=(S_1,T_1)$ of Type 2 such that $T_\omega=T_{\omega_1}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in \boldsymbol{SOL}(f_0)$.

Proof. We take an intersectable system (S,T) of Type 7. First, we assume that $S=\{s_1,s_2,s_3\}$ and $T=\{t_1,t_2,t_3\}$ such that

(1) s_1 is a row and s_2,s_3 are 3×3 blocks,

(2) t_1,t_2,t_3 are columns,

(3) $s_i\cap s_j=\phi$ for $i\neq j, 1\leqq i\leqq 3, 1\leqq j\leqq 3$,

(4) $t_i\cap t_j=\phi$ for $i\neq j, 1\leqq i\leqq 3, 1\leqq j\leqq 3$,

(5) $s_i\cap t_j\neq\phi$ for each i,j, $1\leqq i\leqq 3$, $1\leqq j\leqq 3$.

By (1)–(5) we can choose a set $A_1=\{i_1,i_2,i_3\}\subset J_1$, $A_2=\{i_4,i_5,i_6\}\subset J_1$, $A_3=\{i_7,i_8,i_9\}\subset J_1$ and sets $B_1=\{j_1,j_2,j_3\}\subset J_2$, $B_2=\{j_4,j_5,j_6\}\subset J_2$, $B_3=\{j_7,j_8,j_9\}\subset J_2$ such that

(6) $\{A_1,A_2,A_3\}=\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$,

(7) $\{B_1,B_2,B_3\}=\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$,

(8) $s_1=\{i_1\}\times J_2$,

(9) $s_2=A_2\times B_1$,

(10) $s_3=A_3\times B_1$,

(11) $t_1=J_1\times\{j_1\}$,

(12) $t_2=J_1\times\{j_2\}$,

(13) $t_3=J_1\times\{j_3\}$,

(14) $s=s_1\cup s_2\cup s_3$, $t=t_1\cup t_2\cup t_3$.

We put an intersectable system $\omega_1=(S_1,T_1)$ of Type 2 as follows:

(15) $S_1=\{s_1\}$ and $T_1=\{v_1\}$, where $v_1=A_1\times B_1$.

Take any sudoku matrix $K=(K_\alpha)_{\alpha\in J_1\times J_2}\in \boldsymbol{SMTX}(f,f_0)$ and we put $K'=T_\omega(K)$ $=T(S,T)(K)$ and $K''=T_{\omega_1}(K)=T(S_1,T_1)(K)$. Thus we have

$$(15)\ K'_\alpha=\begin{cases}K_\alpha & \text{for } \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)\\ K_\alpha\cap K_{t-s} & \text{for } \alpha\in s-t\\ K_\alpha\cap K_{s-t} & \text{for } \alpha\in t-s\end{cases},$$

$$(16)\ K''_\alpha=\begin{cases}K_\alpha & \text{for } \alpha\in(J_1\times J_2-s_1\cup v_1)\cup(s_1\cap v_1)\\ K_\alpha\cap K_{v_1-s_1} & \text{for } \alpha\in s_1-v_1\\ K_\alpha\cap K_{s_1-v_1} & \text{for } \alpha\in v_1-s_1\end{cases}.$$

By (1)–(15) we can easily show that

(17) $(J_1\times J_2-s\cup t)\cup(s\cap t)=(J_1\times J_2-s_1\cup v_1)\cup(s_1\cap v_1)$,

(18) $s-t=s_1-v_1$,

(19) $t-s=v_1-s_1$.

By (15)–(19) we have that

(20) $K'_\alpha=K''_\alpha$ for $\alpha\in J_1\times J_2$, i.e.,

(21) $T_\omega=T_{\omega_1}$.

Thus we have (21) for the case of conditions (1) and (2).

By the same way we can show (21) for the other case of Type 7. Hence we have Proposition 48.

Proposition 49. (Type 8) Type 8 has two different types, Type 8A and Type 8B.

(a) For each intersectable system $\omega=(S,T)$ of Type 8A, we have $T_\omega=1$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

(b) For each intersectable system $\omega=(S,T)$ of Type 8B, we have intersectable systems $\omega_1=(S_1,T_1)$ and $\omega_2=(S_2,T_2)$ of Type 2 such that $T_\omega\geqq T_{\omega_1}\cap T_{\omega_2}$ in $\boldsymbol{STRF}(f,f_0)$ for each $f\in\boldsymbol{SOL}(f_0)$.

Proof. We consider an intersectable system $\omega=(S,T)$ of Type 8. For example we consider the following case: We put $S=\{s_1,s_2,s_3\}$ and $T=\{t_1,t_2,t_3\}$ such that

(1) s_1 is a 3×3 block, and s_2,s_3 are rows,

(2) t_1,t_2,t_3 are columns,

(3) $s_i\cap s_j=\phi$ for $i\neq j$, $1\leqq i\leqq 3$, $1\leqq j\leqq 3$,

(4) $t_i\cap t_j=\phi$ for $i\neq j$, $1\leqq i\leqq 3$, $1\leqq j\leqq 3$,

(5) $s_i\cap t_j\neq\phi$ for i,j, $1\leqq i\leqq 3$, $1\leqq j\leqq 3$,

(6) $s=s_1\cup s_2\cup s_3$, $t=t_1\cup t_2\cup t_3$.

By (1)–(5) we can choose sets $A_1=\{i_1,i_2,i_3\}\subset J_1$, $A_2=\{i_4,i_5,i_6\}\subset J_1$, $A_3=\{i_7,i_8,i_9\}\subset J_1$ and sets $B_1=\{j_1,j_2,j_3\}\subset J_2$, $B_2=\{j_4,j_5,j_6\}\subset J_2$, $B_3=\{j_7,j_8,j_9\}\subset J_2$ such that

(7) $\{A_1,A_2,A_3\}=\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$,

(8) $\{B_1,B_2,B_3\}=\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$,

(9) $s_1=A_1\times B_1$,

(10) $s_2=\{i^*\}\times J_2$, $i^*\in A_2\cup A_3$, $i^*\neq i^{**}$,

(11) $s_3=\{i^{**}\}\times J_2$, $i^{**}\in A_2\cup A_3$, $i^*\neq i^{**}$,

(12) $t_1=J_1\times\{j_1\}$,

(13) $t_2=J_1\times\{j_2\}$,

(14) $t_3=J_1\times\{j_3\}$.

(15) $v_2=A_2\times B_1$, $v_3=A_3\times B_1$

Take any sudoku matrix $K=(K_\alpha)_{\alpha\in J_1\times J_2}\in SMTX(f,f_0)$ and we put $K'=T_\omega(K)$ $=T(S,T)(K)$. Thus we have

$$(16)\ K'_\alpha=\begin{cases} K_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t) \\ K_\alpha\cap K_{s-t} & for\ \alpha\in t-s \\ K_\alpha\cap K_{t-s} & for\ \alpha\in s-t \end{cases}.$$

By (10) and (11) we have two cases:

Type 8A: (i) $i^*.i^{**}\in A_2$ or (ii) $i^*,i^{**}\in A_3$.

Type 8B: (iii) $i^*\in A_2$, $i^{**}\in A_3$ or (iv) $i^{**}\in A_2$, $i^*\in A_3$.

We consider Type 8A. We assume the condition (i). We put

(17) $i^*=i_4$ and $i^{**}=i_5$.

Claim 1. $K_{t-s}=J_3$.

Proof of Claim 1. By (1)−(15),(17) we have

(18) $t-s=\{(i_6,j_1),(i_6,j_2),(i_6,j_3)\}\cup v_3$.

By (SMTX) we have

(19) $f(\alpha)\in K_\alpha\subset J_3$ for $\alpha\in J_1\times J_2$.

By (18), (19) we have

(20) $f(v_3)=\{f(\alpha):\alpha\in v_3\}\subset\cup\{K_\alpha:\alpha\in v_3\}=K_{v_3}\subset K_{t-s}\subset J_3$.

Since v_3 is a 3×3 block, by (SDM) we have

(21) $f\mid v_3:v_3\to J_3$ is bijective.

By (21) we have

(22) $f(v_3)=J_3$.

By (20), (22) we have

(23) $K_{t-s}=J_3$.

Thus we have Claim 1.

Claim 2. $K_{s-t}=J_3$.

Proof of Claim 2. We assume that

(24) $K_{s-t}\neq J_3$.

Since $K_{2-t}\subset J_3$, by (24) we have

(25) $J_3-K_{s-t}\neq\phi$.

By (1)−(17) we have

(26) $s-t=(s_2-v_2)\cup(s_3-v_2)$.

By (19), (26) we have

(27) $f(s_2-v_2)\subset K_{s_2-v_2}\subset J_3$,

(28) $f(s_3-v_2)\subset K_{s_3-v_2}\subset J_3$.

By (26), (27), (28) we have

$(J_3-f(s_2-v_2))\cap(J_3-f(s_3-v_2))\supset(J_3-K_{s_2-v_2})\cap(J_3-K_{s_3-v_2})=J_3-K_{s_2-v_2}\cup K_{s_3-v_2}$

$=J_3-K_{(s_2-v_2)\cup(s_3-v_2)}=J_3-K_{s-t}$, i.e.,

(29) $(J_3-f(s_2-v_2))\cap(J_3-f(s_3-v_2))\supset J_3-K_{s-t}$.

Since s_2,s_3 are rows and v_2 is a 3×3 block, by (SDM) we have

(30) $f\mid s_2:s_2\to J_3$ is bijective,

(31) $f\mid s_3:s_3\to J_3$ is bijective,

(32) $f\mid v_2:v_2\to J_3$ is bijective.

By (30),(31) we have

(33) $J_3=f(s_2)=f(s_2-v_2)\cup f(s_2\cap v_2),\ f(s_2-v_2)\cap f(s_2\cap v_2)=\phi$,

(34) $J_3=f(s_3)=f(s_3-v_2)\cup f(s_3\cap v_2),\ f(s_3-v_2)\cap f(s_3\cap v_2)=\phi$.

By (33), (34) we have

(35) $J_3-f(s_2-v_2)=f(s_2\cap v_2)$,

(36) $J_3-f(s_3-v_2)=f(s_3\cap v_2)$.

By (29), (35), (36) we have

(37) $f(s_2\cap v_2)\cap f(s_3\cap v_2)\supset J_3-K_{s-t}$.

Since $J_3-K_{s-t}\neq\phi$ by (25), take any $k_0\in J_3-K_{s-t}$. By (37) there exist α_2, α_3 such that

(38) $\alpha_2\in s_2\cap v_2$, $\alpha_3\in s_3\cap v_3$ and $f(\alpha_2)=k_0=f(\alpha_3)$.

Since $s_2\cap s_3=\phi$ by (3), by (38) we have

(39) $\alpha_2\neq\alpha_3$.

By (38), we have

(40) α_2, $\alpha_3\in v_2$ and $f(\alpha_2)=f(\alpha_3)$.

(39) and (40) mean that $f\mid v_2:v_2\to J_3$ is not bijective. This contradicts to (32).

Hence we have Claim 2.

By (16), Claim 1, Claim 2 we have

$$K'_\alpha=\begin{cases}K_\alpha & for\ \alpha\in(J_1\times J_2-s\cup t)\cup(s\cap t)\\ K_\alpha\cap K_{s-t}=K_\alpha\cap J_3=K_\alpha & for\ \alpha\in t-s\\ K_\alpha\cap K_{t-s}=K_\alpha\cap J_3=K_\alpha & for\ \alpha\in s-t\end{cases}\quad \text{i.e.,}$$

(41) $K'_\alpha=K_\alpha$ for $\alpha\in J_1\times J_2$.

(41) means that

(42) $T_\omega=1$.

Thus, we have the required one for the condition (i). By the same way we can show (42) for the condition (ii). Thus we have (a) for Type 8A.

We consider Type 8B. We assume the condition (iii). We put

(43) $i^{*}=i_4$ and $i^{**}=i_7$.

We put intersectable systems $\omega_1=(S_1,T_1)$, $\omega_2=(S_2,T_2)$ of Type 2 as follows:

(44) $S_1=\{s_2\}, T_1=\{v_2\}$,

(45) $S_2=\{s_3\}, T_2=\{v_3\}$.

We put $K^{*}=T_{\omega_1}(K)=T(S_1,T_1)(K)$. Thus we have

(46) $$K^{*}_{\alpha}=\begin{cases} K_{\alpha} & for\ \alpha\in(J_1\times J_2-s_2\cup v_2)\cup(s_2\cap v_2) \\ K_{\alpha}\cap K_{v_2-s_2} & for\ \alpha\in s_2-v_2 \\ K_{\alpha}\cap K_{s_2-v_2} & for\ \alpha\in v_2-s_2 \end{cases},$$

We put $K^{\#}=T_{\omega_2}(K)=T(S_2,T_2)(K)$. Thus we have

(47) $$K^{\#}_{\alpha}=\begin{cases} K_{\alpha} & for\ \alpha\in(J_1\times J_2-s_3\cup v_3)\cup(s_3\cap v_3) \\ K_{\alpha}\cap K_{v_3-s_3} & for\ \alpha\in s_3-v_3 \\ K_{\alpha}\cap K_{s_3-v_3} & for\ \alpha\in v_3-s_3 \end{cases},$$

We put $K^{!}=K^{*}\cap K^{\#}$.

Claim 3. $K^{!}\leqq K'$.

Proof of Claim 3. By (1)–(15) and (43) we have the decomposition of $J_1\times J_2$ into 5–parts as follows:

(48) $J_1\times J_2=A\cup B\cup C\cup D\cup E$;

(i) $A=(J_1\times J_2-s\cup t)\cup(s\cap t)$,

(ii) $B=s_2-v_2$

(iii) $C=v_2-s_2$,

(iv) $D=s_3-v_3$,

(v) $E=v_3-s_3$

Case (i). Since T_{ω_1} and T_{ω_2} are sudoku transformations by Proposition 42, by Proposition 19 $T_{\omega_1}\cap T_{\omega_2}$ is also a sudoku transformation. Thus we have

(49) $K^{!}=K^{*}\cap K^{\#}=(T_{\omega_1}\cap T_{\omega_2})(K)\leqq K$.

Take any $\alpha\in A$. By (49) we have

(50) $K^{!}_{\alpha}\subset K_{\alpha}$.

By (16)

(51) $K'_\alpha = K_\alpha$.

By (50),(51) we have

(52) $K^!_\alpha \subset K'_\alpha$.

Case (ii). Take any $\alpha \in B$. By (1)−(15) and (43) we have

(53) $\alpha \in s_2 - v_2 \subset s - t$

(54) $\alpha \in (J_1 \times J_2 - s_3 \cup v_3) \cup (s_3 \cap v_3)$,

(55) $v_2 - s_2 \subset t - s$.

By (16),(46),(47),(53),(54) we have

(56) $K'_\alpha = K_\alpha \cap K_{t-s}$,

(57) $K^*_\alpha = K_\alpha \cap K_{v_2 - s_2}$,

(58) $K^\#_\alpha = K_\alpha$.

By (55),(56),(57),(58) we have

$K^!_\alpha = K^*_\alpha \cap K^\#_\alpha = (K_\alpha \cap K_{v_2 - s_2}) \cap K_\alpha = K_\alpha \cap K_{v_2 - s_2} \subset K_\alpha \cap K_{t-s} = K'_\alpha$ i.e.,

(59) $K^!_\alpha \subset K'_\alpha$.

Case (iii). Take any $\alpha \in C$. By (1)−(15) and (43) we have

(60) $\alpha \in v_2 - s_2 \subset t - s$,

(61) $\alpha \in (J_1 \times J_2 - s_3 \cup v_3) \cup (s_3 \cap v_3)$,

(62) $s_2 - v_2 \subset s - t$.

By (16),(46),(47),(60),(61) we have

(63) $K'_\alpha = K_\alpha \cap K_{s-t}$,

(64) $K^*_\alpha = K_\alpha \cap K_{s_2 - v_2}$,

(65) $K^\#_\alpha = K_\alpha$.

By (62),(63),(64),(65) we have

$K^!_\alpha = K^*_\alpha \cap K^\#_\alpha = (K_\alpha \cap K_{s_2 - v_2}) \cap K_\alpha = K_\alpha \cap K_{s_2 - v_2} \subset K_\alpha \cap K_{s-t} = K'_\alpha$ i.e.,

(66) $K^!_\alpha \subset K'_\alpha$.

Case (iv). Take any $\alpha \in D$. By (1)−(15) and (43) we have

(67) $\alpha \in s_3 - v_3 \subset s - t$,

(68) $\alpha \in (J_1 \times J_2 - s_2 \cup v_2) \cup (s_2 \cap v_2)$,

(69) $v_3 - s_3 \subset t - s$.

By (16),(46),(47),(67),(68) we have

(70) $K'_\alpha = K_\alpha \cap K_{t-s}$,

(71) $K^*_\alpha = K_\alpha$,

(72) $K^{\#}_{\alpha}=K_{\alpha}\cap K_{v_3-s_3}$.

By (69),(70),(71),(72) we have

$K^{!}_{\alpha}=K^{*}_{\alpha}\cap K^{\#}_{\alpha}=K_{\alpha}\cap(K_{\alpha}\cap K_{v_3-s_3})=K_{\alpha}\cap K_{v_3-s_3}\subset K_{\alpha}\cap K_{t-s}=K^{'}_{\alpha}$ i.e.,

(73) $K^{!}_{\alpha}\subset K^{'}_{\alpha}$.

Case (v). Take any $\alpha\in D$. By (1)−(15) and (43) we have

(74) $\alpha\in v_3-s_3\subset t-s$,

(75) $\alpha\in(J_1\times J_2-s_2\cup v_2)\cup(s_2\cap v_2)$,

(76) $s_3-v_3\subset s-t$.

By (16),(46),(47),(74),(75) we have

(77) $K^{'}_{\alpha}=K_{\alpha}\cap K_{s-t}$,

(78) $K^{*}_{\alpha}=K_{\alpha}$,

(79) $K^{\#}_{\alpha}=K_{\alpha}\cap K_{s_3-v_3}$.

By (76),(77),(78),(79) we have

$K^{!}_{\alpha}=K^{*}_{\alpha}\cap K^{\#}_{\alpha}=K_{\alpha}\cap(K_{\alpha}\cap K_{s_3-v_3})=K_{\alpha}\cap K_{s_3-v_3}\subset K_{\alpha}\cap K_{s-t}=K^{'}_{\alpha}$ i.e.,

(80) $K^{!}_{\alpha}\subset K^{'}_{\alpha}$.

By (52) in case (i), (59) in case (ii), (66) in case (iii), (73) in case (iv), (80) in case (v) and (48) we can show Claim 3.

Claim 3 means that (b) holds.

Thus, we show Proposition 49 for the conditions (1) and (2). By the same way we can show Proposition 49 for the other conditions. We complete the proof of Proposition 49.

References

[1] T.Kitamoto and T.Watanabe, Matematics and Sudoku I, Bulletin of the Faculty of Education, Yamaguchi University, pp.193−201, vol.64, PT2, 2014.

[2] T.Kitamoto and T.Watanabe, Matematics and Sudoku II, Bulletin of the Faculty of Education, Yamaguchi University, pp.202−208, vol.64, PT2, 2014.

[3] T.Kitamoto and T.Watanabe, Matematics and Sudoku III, Bulletin of the Faculty of Education, Yamaguchi University, pp.79−92, vol.65, PT2, 2015.

[4] T.Kitamoto and T.Watanabe, Matematics and Sudoku IV, Bulletin of the Faculty of Education, Yamaguchi University, pp.93−102, vol.65, PT2, 2015.