Mathematics and Suduku VIII

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We discuss on the worldwide famous Sudoku by using mathematical approach. This paper is the 8th paper in our series, so we use the same notations and terminologies in [1], [2] without any descriptions.

13. A block coordinate system and a block structural system.

In this section we introduce a new useful coordi– nate system in Sudoku.

Ordinally we use the following notations and terminologies as follows:
We alreday use the basic notation $J=\{1,2,3,4,5,6,7,8,9\}$. However, we change it as $J=\{0,1,2,3,4,5,6,7,8\}$. By this change, we does not change our previous arguments and proofs in our Propositions.

Let $Z=\{0,1,2\}$, $Z=Z_1=Z_2$ and $U=Z_2\times Z_1$. We define maps DEC, EXP as follows:

$DEC: J=\{0,1,2,3,4,5,6,7,8\}\to U=Z_2\times Z_1$

$DEC(n)=(n_2,n_1)$ for each $n\in J$

$n=3\times n_2+n_1,\ 0\leqq n_2\leqq 2,\ 0\leqq n_1\leqq 2$

$EXP: U=Z_2\times Z_1\to J=\{0,1,2,3,4,5,6,7,8\}$

$EXP(n_2,n_1)=3\times n_2+n_1$ for each $(n_2,n_1)\in U$.

We can easily show the following:

Proposition 76.
$DEC\circ EXP=1$ and $EXP\circ DEC=1$.

We put $J_1=J_2=J$, $Z_{(1,1)}=Z_{(1,2)}=Z_{(2,1)}=Z_{(2,2)}=Z$. We define maps as follows:

$DEC_1: J_1\to U_1=Z_{(1,2)}\times Z_{(1,1)}$,

$DEC_2: J_2\to U_2=Z_{(2,2)}\times Z_{(2,1)}$,

$DEC_1(i)=(i_2,i_1)$, $i=3\times i_2+i_1,\ 0\leqq i_2\leqq 2,\ 0\leqq i_1\leqq 2$

$DEC_2(j)=(j_2,j_1)$, $j=3\times j_2+j_1,\ 0\leqq j_2\leqq 2,\ 0\leqq j_1\leqq 2$

$EXP_1: U_1=Z_{(1,2)}\times Z_{(1,1)}\to J_1$,

$EXP_2: U_2=Z_{(2,2)}\times Z_{(2,1)}\to J_2$

$EXP_1(i_2,i_1)=3\times i_2+i_1$,

$EXP_2(j_2,j_1)=3\times j_2+j_1$.

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Next we define the map:

$1\times EXC_{(2,3)}\times 1: Z\times Z\times Z\times Z\to Z\times Z\times Z\times Z$

$1\times EXC_{(2,3)}\times 1(t_1,t_2,t_3,t_4)=(t_1,t_3,t_2,t_4)$

for each $(t_1,t_2,t_3,t_4)\in Z\times Z\times Z\times Z$

Now we define $B_2=Z_{(1,2)}\times Z_{(2,2)}$, $B_1=Z_{(1,1)}\times Z_{(2,1)}$ and maps

$BDEC: J_1\times J_2\to B_2\times B_1$,

$BEXP: B_2\times B_1\to J_1\times J_2$ by

$BDEC=(1\times EXC_{(2,3)}\times 1)\circ(DEC_1\times DEC_2)$,

$BEXP=(EXP_1\times EXP_2)\circ(1\times EXC_{(2,3)}\times 1)$.

More clearly we can show that

$BDEC(i,j)=((i_2,j_2),(i_1,j_1))$,

$i=3\times i_2+i_1,\ 0\leqq i_2\leqq 2,\ 0\leqq i_1\leqq 2$

$j=3\times j_2+j_1,\ 0\leqq j_2\leqq 2,\ 0\leqq j_1\leqq 2$

$BEXP((i_2,j_2),(i_1,j_1))=(3\times i_2+i_1, 3\times j_2+j_1)$.

By Proposition 76 we can easily show that

Proposition 77.
$BEXP\circ BDEC=1$ and $BDEC\circ BEXP=1$.

Let $J_4=J_5=J$, we define maps,

$DEC_4: J_4\to B_2=Z_{(1,2)}\times Z_{(2,2)}$,

$DEC_5: J_5\to B_1=Z_{(1,1)}\times Z_{(2,1)}$ by

$DEC_4(u)=(u_2,u_1)$,

$u=3\times u_2+u_1,\ 0\leqq u_2\leqq 2,\ 0\leqq u_1\leqq 2$,

$DEC_5(v)=(v_2,v_1)$,

$v=3\times v_2+v_1,\ 0\leqq v_2\leqq 2,\ 0\leqq v_1\leqq 2$,

$EXP_4: B_2=Z_{(1,2)}\times Z_{(2,2)}\to J_4$,

$EXP_5: B_1=Z_{(1,1)}\times Z_{(2,1)}\to J_5$,

$EXP_4(i_2,j_2)=3\times i_2+j_2$,

$EXP_5(i_1,j_1)=3\times i_1+j_1$.

We define maps:

$DEC^*=DEC_4\times DEC_5: J_4\times J_5\to B_2\times B_1$

$EXP^*=EXP_4\times EXP_5: B_2\times B_1\to J_4\times J_5$.

And also we define maps:

$CDEX=EXP^*\circ BDEC: J_1\times J_2\to J_4\times J_5$

$CEXP=BEXP\circ DEC^*: J_4\times J_5\to J_1\times J_2$.

Let $TRS: X\times Y\to Y\times X$ be a trasposed map.

$TRS(x,y)=(y,x)$ for each $(x,y)\in X\times Y$.

We define maps

$CTDEX=TRS\circ CDEC\circ TRS: J_2\times J_1\to J_5\times J_4$

$CTEXP=TRS\circ CEXP\circ TRS: J_5\times J_4\to J_2\times J_1$.

More clearly we can show that

$CDEC(i,j)=(3\times i_2+j_2, 3\times i_1+j_1)$

$CEXP(u,v)=(3\times u_2+v_2, 3\times u_1+v_1)$

$CTDEC(i,j)=(3\times j_1+i_1, 3\times j_2+i_2)$

$CTEXP(u,v)=(3\times v_1+u_1, 3\times v_2+u_2)$

for $i,j,u,v\in J$, $i=3\times i_2+i_1$, $j=3\times j_2+j_1$, $u=3\times u_2+u_1$, $v=3\times v_2+v_1$.

By definitions, we can easily show that

Proposition 78.

(a) $EXP^*\circ DEC^*=1$ and $DEC^*\circ EXP^*=1$.

(b) $CEXP\circ CDEC=1$ and $CDEC\circ CEXP=1$.

(c) $CEXP=CDEC$.

(d) $CTEXP\circ CTDEC=1$, $CTDEC\circ CTEXP=1$.

(e) $CTEXP=CTDEC$.

We say that $J_4\times J_5$ is a block coordinate system, and $B_2\times B_1$ is a block structual system. $B_2\times B_1$ has more rich mathematical structures than $J_4\times J_5$.

Usually, we use the following notations:
Let $I(0)=\{0,1,2\}$, $I(1)=\{3,4,5\}$, $I(2)=\{6,7,8\}$,

$B(0)=I(0)\times I(0)$, $B(1)=I(0)\times I(1)$, $B(2)=I(0)\times I(2)$,

$B(3)=I(1)\times I(0)$, $B(4)=I(1)\times I(1)$, $B(5)=I(1)\times I(2)$,

$B(6)=I(2)\times I(0)$, $B(7)=I(2)\times I(1)$, $B(8)=I(2)\times I(2)$.

These notations are formulated as following simple rules:

$I(i_2)=\{3\times i_2+0, 3\times i_2+1, 3\times i_2+2\}$, for each $0\leqq i_2\leqq 2$,

$I(i_2)\times I(j_2)=B(3\times i_2+j_2)$ for each $b_2=(i_2,j_2)\in B_2$.

Therefore, we use the notation:

$I(b_2)=I(i_2)\times I(j_2)=B(EXP(b_2))$ for each $b_2=(i_2,j_2)\in B_2$.

Proposition 79.

Let $i,j,s\in J$, $DEC(i)=(i_2,i_1)$, $DEC(j)=(j_2,j_1)$, $DEC(s)=(s_2,s_1)$, $b_2=(i_2,j_2)\in B_2$. We have that

(a) $DEC(I(i_2))=\{i_2\}\times Z_1$, $DEC(J)=Z_2\times Z_1$,

$EXP(\{i_2\}\times Z_1)=I(i_2)$,

$EXP(Z_2\times\{i_1\})=\{i_1, i_1+3, i_1+6\}$,

$EXP(Z_2\times Z_1)=J$.

(b) $BDEC(I(b_2))=\{b_2\}\times B_1$,

$BDEC(\{i\}\times J_2)=\{i_2\}\times Z_{(2,2)}\times\{i_1\}\times Z_{(1,1)}$,

$BDEC(J_1\times\{j\})=Z_{(1,2)}\times\{j_2\}\times Z_{(1,1)}\times\{j_1\}$,

(c) $CDEC(B(s))=\{s\}\times J_5$

$CDEC(\{i\}\times J_2)=B(i)$.

$CDEC(J_1\times\{j\})=\{j_2,j_2+3,j_2+6\}\times\{j_1,j_1+3,j_1+6\}$.

(d) $CTDEC(B(s))=J_4\times\{3\times s_1+s_2\}$,

$CTDEC(\{i\}\times J_2)=\{i_1,i_1+3,i_1+6\}\times\{i_2,i_2+3,i_2+6\}$

$CTDEC(J_1\times\{j\})=B(3\times j_1+j_2)$.

Proof.

We show (a).

$DEC(I(i_2))$

$=DEC(\{3\times i_2+0, 3\times i_2+1, 3\times i_2+2\})$

$=\{DEC(3\times i_2), DEC(3\times i_2+1), DEC(3\times i_2+2)\}$

$=\{(i_2,0), (i_2,1), (i_2,2)\}=\{i_2\}\times Z_1$,

$DEC(J)=DEC(I(0)\cup I(1)\cup I(2))$

$=DEC(I(0))\cup DEC(I(1))\cup DEC(I(2))$

$=\{0\}\times Z_1\cup\{1\}\times Z_1\cup\{2\}\times Z_1=Z_2\times Z_1$,

$EXP(\{i_2\}\times Z_1)=EXP(\{(i_2,0), (i_2,1), (i_2,2)\})$

$=\{3\times i_2+0, 3\times i_2+1, 3\times i_2+2\}=I(i_2)$,

$EXP\{Z_2\times\{i_1\}\}=EXP(\{(0,i_1), (1,i_1), (2,i_1)\})$

$=\{3\times 0+i_1, 3\times 1+i_1, 3\times 2+i_1\}=\{i_1,i_1+3,i_1+6\}$,

$EXP(Z_2\times Z_1)=EXP(\{0\}\times Z_1\cup\{1\}\times Z_1\cup\{2\}\times Z_1)$

$=I(0)\cup I(1)\cup I(2)=J$.

We show (b).

By (a) we have that

$BDEC(I(b_2))=BDEC(I(i_2)\times I(j_2))$

$=(1\times EXC_{(2,3)}\times 1)(DEC(I(i_2))\times DEC(I(j_2)))$

$=(1\times EXC_{(2,3)}\times 1)(\{i_2\}\times Z_{(1,1)}\times\{j_2\}\times Z_{(2,1)})$

$=\{i_2\}\times\{j_2\}\times Z_{(1,1)}\times Z_{(2,1)}$

$=\{(i_2,j_2)\}\times Z_{(1,1)}\times Z_{(2,1)}=\{b_2\}\times B_1$,

$BDEC(\{i\}\times I(j_2))$

$=(1\times EXC_{(2,3)}\times 1)(DEC(i), DEC(I(j_2)))$

$=(1\times EXC_{(2,3)}\times 1)(\{(i_2,i_1)\}\times\{j_2\}\times Z_{(2,1)})$

$=\{(i_2,j_2)\}\times\{i_1\}\times Z_{(2,1)}=\{i_2\}\times\{j_2\}\times\{i_1\}\times Z_{(2,1)}$.

Since $J_2=I(0)\cup I(1)\cup I(2)$, by (2) we have that

$BDEC(\{i\}\times J_2)$

$=BDEC(\{i\}\times I(0)\cup\{i\}\times I(1)\cup\{i\}\times I(2))$

$=BDEC(\{i\}\times I(0))\cup BDEC(\{i\}\times I(1))$
$\cup DEC(\{i\}\times I(2))$

$=\{(i_2,0)\}\times\{i_1\}\times Z_{(2,1)}\cup\{(i_2,1)\}\times\{i_1\}\times Z_{(2,1)}$
$\cup\{(i_2,2)\}\times\{i_1\}\times Z_{(2,1)}$

$=\{(i_2,0), (i_2,1), (i_2,2)\}\times\{i_1\}\times Z_{(2,1)}$

$=\{i_2\}\times Z_{(2,2)}\times\{i_1\}\times Z_{(2,1)}$.

$BDEC(I(i_2)\times\{j\})$

$=(1\times EXC_{(2,3)}\times 1)(DEC(I(i_2)), DEC(\{j\}))$

$= (1 \times EXC_{(2,3)} \times 1)(\{i_2\} \times Z_{(1,1)} \times \{(j_2, j_1)\})$
$= \{i_2\} \times \{j_2\} \times Z_{(1,1)} \times \{j_1\} = \{(i_2, j_2)\} \times Z_{(1,1)} \times \{j_1\}.$

Since $J_1 = I(0) \cup I(1) \cup I(2)$, we have
$BDEC(J_1 \times \{j\})$
$= BDEC(I(0) \times \{j\}) \cup BDEC(I(1) \times \{j\})$
$\cup BDEC(I(2) \times \{j\})$
$= \{(0, j_2)\} \times Z_{(1,1)} \times \{j_1\} \cup \{(1, j_2)\} \times Z_{(1,1)} \times \{j_1\}$
$\cup \{(2, j_2)\} \times Z_{(1,1)} \times \{j_1\}$
$= \{(0, j_2), (1, j_2), (2, j_2)\} \times Z_{(1,1)} \times \{j_1\}$
$= Z_{(1,2)} \times \{j_2\} \times Z_{(1,1)} \times \{j_1\}.$

We show (c).
We can take $b_{20} = (s_2, s_1) \in B_2$ and $EXP(b_{20}) = s$. By (b) we have that
$BDEC(I(b_{20})) = \{b_{20}\} \times B_1.$
By Proposition 79, we have that
$CDEC(B(s)) = CDEC(I(b_{20}))$
$= (EXP \times EXP) \circ BDEC(I(b_{20}))$
$= (EXP \times EXP)(\{b_{20}\} \times B_1)$
$= \{EXP(b_{20})\} \times EXP(B_1) = \{s\} \times J_5.$

$CDEC(\{i\} \times J_2) = (EXP \times EXP) \circ BDEC(\{i\} \times J_2)$
$= (EXP \times EXP)(\{i_2\} \times Z_{(2,2)} \times \{i_1\} \times Z_{(1,1)})$
$= EXP(\{i_2\} \times Z_{(2,2)}) \times EXP(\{i_1\} \times Z_{(1,1)})$
$= I(i_2) \times I(i_1) = B(3 \times i_2 + i_1) = B(i).$

$CDEC(J_1 \times \{j\}) = (EXP \times EXP) \circ BDEC(J_1 \times \{j\})$
$= (EXP \times EXP)(Z_{(1,2)} \times \{j_2\} \times Z_{(1,1)} \times \{j_1\})$
$= EXP(Z_{(1,2)} \times \{j_2\}) \times EXP(Z_{(1,1)} \times \{j_1\})$
$= \{j_2, j_2+3, j_2+6\} \times \{j_1, j_1+3, j_1+6\}.$

We show (d).
$CTDEX(B(s)) = TRS \circ CDEC \circ TRS(I(s_2) \times I(s_1))$
$= TRS \circ CDEC(I(s_1) \times I(s_2))$
$= TRS \circ CDEC(B(3 \times s_1 + s_2))$
$= TRS(\{3 \times s_1 + s_2\} \times J_5) = J_4 \times \{3 \times s_1 + s_2\}.$

$CTDEC(\{i\} \times J_2) = TRS \circ CDEC \circ TRS(\{i\} \times J_2)$
$= TRS \circ CDED(J_1 \times \{i\})$
$= TRS(\{i_2, i_2+3, i_2+6\} \times \{i_1, i_1+3, i_1+6\})$
$= \{i_1, i_1+3, i_1+6\} \times \{i_2, i_2+3, i_2+6\}.$

$CTDEX(J_1 \times \{j\}) = TRS \circ CDEC \circ TRS(J_1 \times \{j\})$
$= TRS \circ CDEC(\{j\} \times J_2) = TRS(B(j))$
$= TRS(I(j_2) \times I(j_1)) = I(j_1) \times I(j_2) = B(3 \times j_1 + j_2).$

By Proposition 79 we have the following:

Proposition 80.

(a) $CDEC: J_1 \times J_2 \to J_4 \times J_5$ has the properties:
(i) All rows are transformed into blocks.
(ii) All blocks are trnsformed into rows.
(b) $CTDEC: J_1 \times J_2 \to J_4 \times J_5$ has the properties:
(i) All columns are transformed into blocks.
(ii) All blocks are trnsformed into columns.

14. Basic sets.

Let $T \subset J_1 \times J_2$. A set T is a basic set provided that it satisfies the following condition:
(BA) $|T \cap S| = 1$ for each $S \in BLK$.
Then, we put that $T \cap S = \{t(S)\}$ for each $S \in BLK$.
Here, $BLK = rOW \cup cOL \cup bLK$, which is the collection of all rows, all columns and all 3×3 blocks in $J_1 \times J_2$.
Now, for each $b_2 = (i_2, j_2) \in B_2$ we put $I(b_2) = I(i_2) \times I(j_2) \subset J_1 \times J_2$. Then $bLK = \{I(b_2) : b_2 \in B_2\}$.

Proposition 81.
If T is a basic set, then we have that
(i) $T = \{t(S) : S \in bLK\}$, where $T \cap S = \{t(S)\}$ for each $S \in bLK$.
(ii) $|T| = 9$.
Proof. Since we have 9 blocks, by (BA) for each block S, $|S \cap T| = 1$. Thus
(1) $S \cap T = \{t(S)\}$ for each block $S \in bLK$.

Claim 1.
$T = \{t(S) : S \in bLK\}$
For each block S, by (1) we have that
(2) $\{t(S) : S \in bLK\} \subset T$.

We show that
(3) $T \subset \{t(S) : S \in bLK\}$.
We assume that (3) does not hold. Then we have a $t_0 \in T$ such that
(4) $t_0 \in T - \{t(S) : S \in bLK\}$.
Since $t_0 \in T \subset J_1 \times J_2$, there is a block $S_0 \in bLK$ such that
(5) $t_0 \in S_0$.
By (1), we have that
(6) $t(S_0) \in S_0 \cap T$.
By (5), (6) we have that
(7) $t_0, t(S_0) \in S_0 \cap T$.
By (BA) and (7) we have that
(8) $t_0 = t(S_0)$.
By (4) we have that $t_0 \notin \{t(S) : S \in bLK\}$, thus
(9) $t_0 \neq t(S_0)$.
(9) contradicts to (8). Hence we have (3).
By (2) and (3) we have Claim 1.

We show that
(10) $t(S_1) \neq t(S_2)$ for blocks $S_1, S_2 \in bLK$ with $S_1 \neq S_2$.

We assume that (10) does not hold.
Then we have blocks $S_1, S_2 \in bLK$ such that
(11) $S_1 \neq S_2$ and
(12) $t(S_1) = t(S_2)$.
By (1), (12) we have that $S_1 \ni t(S_1) = t(S_1) \in S_2$. Thus
(13) $S_1 \cap S_2 \neq \phi$.
By (11) we have that
(14) $S_1 \cap S_2 = \phi$.
(14) contradicts to (13). Hence, (10) does hold.

By (10) we have that
(15) $|\{t(S): S \in bLK\}| = |bLK| = 9$.
By (15) and Claim 1 we have that
(16) $|T| = 9$.
Hence, we have Proposition 81.

Let $B = B_2 \times B_1$ be the block structual system. Here $B_1 = Z_{(1,1)} \times Z_{(2,1)}$ and $BDEC: J_1 \times J_2 \to B$, $BEXP: B \to J_1 \times J_2$ are the block decomposition map, the block expansion map, respectively.

Proposition 82.
Let T be a basic set. There is the unique map $\psi: B_2 \to B_1$ such that
$T \cap (I(i_2) \times I(j_2)) = \{BEXP(b_2, \psi(b_2))\}$
for each $b_2 = (i_2, j_2) \in B_2$.
Proof. Take any $b_2 = (i_2, j_2) \in B_2 = Z_{(1,2)} \times Z_{(2,2)}$. Let $I(b_2) = I(i_2) \times I(j_2)$ be the block in $J_1 \times J_2$. We put $T \cap T(b_2) = \{t(b_2)\}$. Thus
(1) $T \cap I(b_2) = T \cap (I(i_2) \times I(j_2)) = \{t(i_2, j_2)\} = \{t(b_2)\}$ for each $b_2 = (i_2, j_2) \in bLK$.
Since $t(b_2) \in J_1 \times J_2$, we put
(2) $t(b_2) = (i(b_2), j(b_2))$, $0 \leqq i(b_2) \leqq 8$, $0 \leqq j(b_2) \leqq 8$.
Since $t(b_2) = (i(b_2), j(b_2)) \in I(i_2) \times I(j_2)$, we have
(3) $i(b_2) \in I(i_2)$, $j(b_2) \in I(j_2)$.
Since $I(i_2) = \{3 \times i_2, 3 \times i_2 + 1, 3 \times i_2 + 2\}$ and $I(j_2) = \{3 \times j_2, 3 \times j_2 + 1, 3 \times j_2 + 2\}$, by (3) we have the following decompositions:
(4) $i(b_2) = 3 \times i_2 + i_1(b_2)$, $0 \leqq i_1(b_2) \leqq 2$
(5) $j(b_2) = 3 \times j_2 + j_1(b_2)$, $0 \leqq j_1(b_2) \leqq 2$.
By (4) and (5) we have that
(6) $DEC(i(b_2)) = (i_2, i_1(b_2))$, and
(7) $DEC(j(b_2)) = (j_2, j_1(b_2))$.
By (6) and (7) we have that
(8) $BDEC(t(b_2)) = ((i_2, j_2), (i_1(b_2), j_1(b_2))) \in B_2 \times B_1$.
We put $t_1(b_2) = (i_1(b_2), j_1(b_2)) \in B_1$. By (8) we have
(9) $BDEC(t(b_2)) = (b_2, t_1(b_2))$.
By (9) and Proposition 76 we have that
(10) $BEXP(b_2, t_1(b_2)) = t(b_2)$.

Therefore, we can define a map $\psi: B_2 \to B_1$ by
(11) $\psi(b_2) = t_1(b_2)$ for each $b_2 \in B_2$.
By (1), (10), (11) we have that
(12) $T \cap (I(i_2) \times I(j_2)) = \{BEXP(b_2, \psi(b_2))\}$ for each $b_2 = (i_2, j_2) \in B_2$.
Thus, by (12), our map ψ has the required property.

Next, we show the uniqueness of ψ.
Let $\Omega: B_2 \to B_1$ be a map with
(13) $T \cap (I(i_2) \times I(j_2)) = \{BEXP(b_2, \Omega(b_2))\}$ for each $b_2 = (i_2, j_2) \in B_2$.

Claim 1.
$\psi = \Omega$.
Take any $b_2 = (i_2, j_2) \in B_2$. By (12) and (13) we have that
(14) $BEXP(b_2, \psi(b_2)) = BEXP(b_2, \Omega(b_2))$.
By (14) we have that
(15) $BDEC(BEXP(b_2, \psi(b_2)))$
$= BDEC(BEXP(b_2, \Omega(b_2)))$.
By Proposition 8—3 we have that
(16) $BDEC(BEXP(b_2, \psi(b_2)))$,
$= BDEC \circ BEXP(b_2, \psi(b_2)) = (b_2 . \psi(b_2))$,
(17) $BDEC(BEXP(b_2, \Omega(b_2)))$
$= BDEC \circ BEXP(b_2, \Omega(b_2)) = (b_2 . \Omega(b_2))$.
By (15), (16), (17) we have that
(18) $(b_2, \psi(b_2)) = (b_2, \Omega(b_2))$.
By (18) we have that
(19) $\psi(b_2) = \Omega(b_2)$.
Then, by (19)
(20) $\psi = \Omega$.
Hence, we have Claim 1.
Therefore, we show Proposition 82.

Let $proj_X: X \times Y \to X$ and $proj_Y: X \times Y \to Y$ be projections. That is,
$proj_X(x, y) = x$ and $proj_Y(x, y) = y$ for each $(x, y) \in X \times Y$.

Proposition 83.
Let T be a basic set. There are unique bijective maps
$\pi_{i_2}: Z_{(2,2)} \to Z_{(1,1)}$ for each $i_2 \in Z_{(1,2)}$
such that
$proj_{J_1} \circ BEXP(b_2, \psi(b_2)) = EXP(i_2, \pi_{i_2}(j_2))$
for each $b_2 = (i_2, j_2) \in B_2$.
Proof. We use the same notations and statements

(1)−(20) in the proof of Proposition 82.

For each $i_2 \in Z_{(1,2)}$ we define a map

$\pi_{i_2}: Z_{(2,2)} \to Z_{(1,1)}$

as follows:

(21) $\pi_{i_2}(j_2) = proj_{Z_{(1,1)}}(\psi(i_2, j_2))$ for each $j_2 \in Z_{(2,2)}$.

Since $\psi(b_2) = t_1(b_2) = (i_1(b_2), j_1(b_2))$ by (11), we have that

(22) $proj_{Z_{(1,1)}}(\psi(b_2)) = i_1(b_2)$ for each $b_2 = (i_2, j_2) \in B_2$.

By (21), (22) we have that

(23) $\pi_{i_2}(j_2) = i_1(i_2, j_2)$

By (4), (23) we have that

(24) $EXP(i_2, \pi_{i_2}(j_2)) = 3 \times i_2 + \pi_{i_2}(j_2)$.

$= 3 \times i_2 + i_1(i_2, j_2) = i(i_2, j_2)$.

By (2), (10), (11) we have that

(25) $proj_{J_1}(BEXP(b_2, \psi(b_2)))$

$= proj_{J_1}(BEXP(b_2, t_1(b_2))) = proj_{J_1} t(b_2)$

$= proj_{J_1}(i(b_2), j(b_2)) = i(b_2) = i(i_2, j_2)$.

By (24), (25) we have that

(26) $proj_{J_1}(BEXP(b_2, \psi(b_2))) = EXP(i_2, \pi_{i_2}(j_2))$.

Thus, π_{i_2} has the required property.

Claim 2.

$\pi_{i_2}: Z_{(2,2)} \to Z_{(1,1)}$ is injective for each $i_2 \in Z_{(1,2)}$.

We assume that Claim 2 does not hold. Then we have an $i_2 \in Z_{(1,2)}$ such that

(27) $\pi_{i_2}: Z_{(2,2)} \to Z_{(1,1)}$ is not injective.

By (27) we have $j_2, j_2^* \in Z_{(2,2)}$ such that

(28) $j_2 \neq j_2^*$ and

(29) $\pi_{i_2}(j_2) = \pi_{i_2}(j_2^*)$.

By (28) we have that

(30) $I(j_2) \cap I(j_2^*) = \phi$.

By (30) we have that

(31) $I(i_2) \times I(j_2) \cap I(i_2) \times I(j_2^*) = \phi$.

By (31) we have that

(32) $(T \cap (I(i_2) \times I(j_2))) \cap (T \cap (I(i_2) \times I(j_2^*))) = \phi$.

We put $b_2 = (i_2, j_2)$, $b_2^* = (i_2, j_2^*) \in B_2$. By (1), (32) we have that

(33) $\{t(b_2)\} \cap \{t(b_2^*)\} = \phi$.

By (33) we have that

(34) $t(b_2) \neq t(b_2^*)$.

By (29) we have that

(35) $EXP(i_2, \pi_{i_2}(j_2)) = EXP(i_2, \pi_{i_2}(j_2^*))$.

By (26), (35) we have that

(36) $proj_{J_1}(BEXP(b_2, \psi(b_2)))$

$= proj_{J_1}(BEXP(b_2^*, \psi(b_2^*)))$.

By (10), (36) we have that

(37) $proj_{J_1}(t(b_2)) = proj_{J_1}(t(b_2^*))$.

By (2) we have that

(38) $t(b_2) = (i(b_2), j(b_2))$, $t(b_2^*) = (i(b_2^*), j(b_2^*))$.

By (37), (38) we have that

(39) $i(b_2) = i(b_2^*) = u$.

Let $r(u) = \{(u, j): j \in J_2\}$ be the u−th row in $J_1 \times J_2$. By (38), (39) we have that

(40) $t(b_2), t(b_2^*) \in r(u)$.

By (1) we have that

(41) $t(b_2), t(b_2^*) \in T$.

By (40),(41) we have that

(42) $t(b_2), t(b_2^*) \in r(u) \cap T$.

By (BA) we have that

(43) $| r(u) \cap T | = 1$.

By (42), (43) we have that

(44) $t(b_2) = t(b_2^*)$.

(44) contradicts to (34). Hence, we have Claim 2

Claim 3.

$\pi_{i_2}: Z_{(2,2)} \to Z_{(1,1)}$ is surjective for each $i_2 \in Z_{(1,2)}$.

Take any $i_2 \in Z_{(1,2)}$ and any $v_1 \in Z_{(1,1)}$. We put

(45) $v = 3 \times i_2 + v_1 \in J_1$.

Let $r(v) = \{(v, j): j \in J_2\}$ be the v−th row of $J_1 \times J_2$. By (BA) we have a $t_v \in J_1 \times J_2$ such that

(46) $r(v) \cap T = \{t_v\}$.

Since $t_v \in r(v)$ by (46), we can put

(47) $t_v = (v, w)$.

Since $w \in J_2$, we have the decomposition

(48) $w = 3 \times w_2 + w_1$, $0 \leqq w_2 \leqq 2$, $0 \leqq w_1 \leqq 2$.

By (45), (47),(48) we have that

(49) $t_v = (v, w) \in I(i_2) \times I(w_2)$.

Since $t_v \in T$ by (46), by (49) we have that

(50) $t_v \in T \cap (I(i_2) \times I(w_2))$.

Let $b_2^{**} = (i_2, w_2) \in B_2$. By (1) we have that

(51) $T \cap (I(i_2) \times I(w_2)) = \{t(b_2^{**})\}$.

By (50), (51) we have that

(52) $t_v = t(b_2^{**})$.

By (52) we have that

(53) $v = i(b_2^{**})$,

(54) $w = j(b_2^{**})$.

By (45), (53) we have that

(55) $v_1 = i_1(b_2^{**})$.

By (48), (54) we have that

(56) $w_1 = j_1(b_2^{**})$.

By (11), (55), (56) we have that

(57) $\psi(b_2^{**})=(v_1, w_1)$.

Thus, by (21), (57) we have that

(58) $\pi_{i_2}(w_2)=proj_{Z_{(1,1)}}\psi(i_2, w_2)=proj_{Z_{(1,1)}}\psi(b_2^{**})$

$=proj_{Z_{(1,1)}}(v_1,w_1)=v_1$.

Then, by (58), we have that π_{i_2} is surjective.

Hence, we have Claim 3.

Claim 4.

The uniquess of π_{i_2} for each $i_2\in Z_{(1,2)}$.

Take any $i_2\in Z_{(1,2)}$. We take any map $\pi^*_{i_2}:Z_{(2,2)}\to Z_{(1,1)}$ such that

(59) $proj_{J_1}\circ BEXP((b_2), \psi(b_2))=EXP(i_2, \pi^*_{i_2}(j_2))$ for each $b_2=(i_2,j_2)\in B_2$.

Take any $j_2\in Z_{(2,2)}$ and put $b_2=(i_2,j_2)\in B_2$. We put

(60) $n=proj_{J_1}\circ BEXP((b_2), \psi(b_2))$.

By (26), (59), (60) we have that

(61) $n=EXP(i_2, \pi_{i_2}(j_2))=3\times i_2+\pi_{i_2}(j_2)$,

(62) $n=EXP(i_2, \pi^*_{i_2}(j_2))=3\times i_2+\pi^*_{i_2}(j_2)$.

Since $\pi_{i_2}(j_2), \pi^*_{i_2}(j_2)\in Z_{(1,1)}$, we have that

(63) $0\leqq\pi_{i_2}(j_2)\leqq 2$, $0\leqq\pi^*_{i_2}(j_2)\leqq 2$.

By (61), (62), (63) we have that

(64) $\pi_{i_2}(j_2)=\pi^*_{i_2}(j_2)$.

By (64) we have that

(65) $\pi_{i_2}=\pi^*_{i_2}$.

Hence, by (65) we have Claim 4.

Therefore, by Claim 1, Claim 2, Claim 3, Claim 4, we have Proposition 83.

For each $j_2\in Z_{(2,2)}$, we can define a map $\rho_{j_2}:Z_{(1,2)}\to Z_{(2,1)}$ by

$\rho_{j_2}(i_2)=proj_{Z_{(2,1)}}(\psi(i_2,j_2))$ for each $i_2\in Z_{(1,2)}$.

By the similar way as the proof of Proposition 83, we can show Proposition 84.

Proposition 84.

Let T be a basic set. There are unique bijective maps $\rho_{j_2}:Z_{(1,2)}\to Z_{(2,1)}$ for each $j_2\in Z_{(2,2)}$ such that

$proj_{J_2}\circ BEXP(b_2,\psi(b_2))=EXP(j_2, \rho_{j_2}(i_2))$ for each $b_2=(i_2,j_2)\in B_2$.

We define maps $\xi_1:B_2\to J_1$ and $\xi_2:B_2\to J_2$ by

$\xi_1(b_2)=EXP(i_2, \pi_{i_2}(j_2))$ for $b_2=(i_2,j_2)\in B_2$,

$\xi_2(b_2)=EXP(j_2, \rho_{j_2}(i_2))$ for $b_2=(i_2,j_2)\in B_2$.

And we define block−graphs of ξ_1 and ξ_2 by

$BGRP(\xi_1)=\cup\{\{\xi_1(i_2,j_2)\}\times I(j_2):\ b_2=(i_2,j_2)\in B_2\}\subset J_1\times J_2$,

$BGRP(\xi_2)=\cup\{I(i_2)\times\{\xi_2(i_2,j_2)\}:\ b_2=(i_2,j_2)\in B_2\}\subset J_1\times J_2$.

Proposition 85.

$I(b_2)\cap BGRP(\xi_1)=\{\xi_1(b_2)\}\times I(j_2)$ for each $b_2=(i_2,j_2)\in B_2$

Proof. Take any $b_{20}=(i_{20},j_{20})\in B_2$. First, we show Claim 1:

Claim 1.

$I(b_{20})\cap BGRP(\xi_1)\subset\{\xi_1(b_{20})\}\times I(j_{20})$.

Take any point p such that

(1) $p=(p_1,p_2)\in I(b_{20})\cap BGRP(\xi_1)$.

Thus, we have

(2) $p=(p_1, p_2)\in I(b_{20})=I(i_{20})\times I(j_{20})$, and

(3) $p=(p_1,p_2)\in BGRP(\xi_1)$

$=\cup\{\{\xi_1(i_2,j_2)\}\times I(j_2):\ b_2=(i_2,j_2)\in B_2\}$.

By (2) we have that

(4) $p_1\in I(i_{20})$

(5) $p_2\in I(j_{20})$.

By (3) we have a $b_{21}=(i_{21},j_{21})\in B_2$ such that

(6) $p=(p_1,p_2)\in\{\xi_1(i_{21},j_{21})\}\times I(j_{21})$.

By (6) we have that

(7) $p_1=\xi_1(i_{21},j_{21})$,

(8) $p_2\in I(j_{21})$.

By (5), (8) we have that

(9) $I(j_{20})\cap I(j_{21})\neq\phi$.

By (9) we have that

(10) $j_{20}=j_{21}$.

By (7) we have that

(11) $p_1=\xi_1(i_{21},j_{21})=EXP(i_{21}, \pi_{i_{21}}(j_{21}))$

$=3\times i_{21}+\pi_{i_{21}}(j_{21})\in I(i_{21})$.

By (4), (11) we have that

(12) $I(i_{20})\cap I(i_{21})\neq\phi$.

By (12) we have that

(13) $i_{20}=i_{21}$.

By (10), (13) we have that

(14) $b_{21}=(i_{21},j_{21})=(i_{20}, j_{20})=b_{20}$.

By (11), (14) we have that

(15) $p_1=\xi_1(b_{21})=\xi_1(b_{20})$.

By (5), (15) we have that

(16) $p=(p_1, p_2)\in\{\xi_1(b_{20})\}\times I(j_{20})$.

By (16) we have Claim 1.

Claim 2.

$I(b_{20})\cap BGRP(\xi_1)\supset\{\xi_1(b_{20})\}\times I(j_{20})$

By the definitions we have that

(17) $\xi_1(b_{20})=EXP(i_{20}, \pi_{i_{20}}(j_{20}))$

$= 3 \times i_{20} + \pi_{i_{20}}(j_{20}) \in I(i_{20})$.

By (17) we have that

(18) $\{\xi_1(b_{20})\} \times I(j_{20}) \subset I(i_{20}) \times I(j_{20}) = I(b_{20})$.

By the definitions, we have

(19) $\{\xi_1(b_{20})\} \times I(j_{20}) \subset BGRP(\xi_1)$.

By (18), (19) we have Claim 2.

Hence by Claim 1 and Claim 2 we have Proposition 85.

By the similar way as Proposition 85, we can show the following Proposition 86.

Proposition 86.

$I(b_2) \cap BGRP(\xi_2) = I(i_2) \times \{\xi_2(b_2)\}$ for each $b_2 = (i_2, j_2) \in B_2$.

Proposition 87.

(a) $I(b_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2) = \{(\xi_1(b_2), \xi_2(b_2))\}$ for each $b_2 = (i_2, j_2) \in B_2$.

(b) $BGRP(\xi_1) \cap BGRP(\xi_2) = \{(\xi_1(b_2), \xi_2(b_2)) : b_2 \in B_2\}$

Proof. First, we show (a).

Take any $b_2 = (i_2, j_2) \in B_2$. By Proposition 85, Proposition 86 we have that

$I(b_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2)$

$= (I(b_2) \cap BGRP(\xi_1)) \cap (I(b_2) \cap BGRP(\xi_2))$

$= (\{\xi_1(b_2)\} \times I(j_2)) \cap (I(i_2) \times \{\xi_2(b_2)\})$

$= \{(\xi_1(b_2), \xi_2(b_2))\}$.

Thus, we have (a).

Next, we show (b).

Since $J_1 \times J_2 = \cup\{I(b_2) : b_2 \in B_2\}$, we have that

$BGRP(\xi_1) \cap BGRP(\xi_2)$

$= (J_1 \times J_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2)$

$= (\cup\{I(b_2) : b_2 \in B_2\}) \cap BGRP(\xi_1) \cap BGRP(\xi_2)$

$= \cup\{I(b_2) \cap BGRP(\xi_1) \cap BGRP(\xi_2) : b_2 \in B_2\}$

$= \{(\xi_1(b_2), \xi_2(b_2)) : b_2 \in B_2\}$.

Thus, we have (b).

Hence, we have Proposition 87.

Proposition 88.

$BGRP(\xi_1) \cap BGRP(\xi_2) = T$.

Proof. Take any $b_2 = (i_2, j_2) \in B_2$

By Propositions 83 and 84, we have

(1) $proj_{J_1}(BEXP(b_2, \psi(b_2)))$

$= EXP(i_2, \pi_{i_2}(j_2)) = \xi_1(b_2)$

(2) $proj_{J_2}(BEXP(b_2, \psi(b_2)))$

$= EXP(j_2, \rho_{j_2}(i_2)) = \xi_2(b_2)$.

By (1), (2) we have that

(3) $BEXP(b_2, \psi(b_2)) = (\xi_1(b_2), \xi_2(b_2))$.

By Proposition 82

(4) $T \cap I(b_2) = \{BEXP(b_2, \psi(b_2))\}$.

By Proposition 81 we have that

(5) $T \cap I(b_2) = \{t(I(b_2))\}$.

By (4) and (5) we have that

(6) $t(I(b_2)) = BEXP(b_2, \psi(b_2))$.

Thus by (3), (6), Propositions 81 and 87, we have that

(7) $T = \{t(I(b_2)) : I(b_2) \in bLK\}$

$= \{BEXP(b_2, \psi(b_2)) : b_2 \in B_2\}$

$= \{(\xi_1(b_2), \xi_2(b_2)) : b_2 \in B_2\}$

$= BGRP(\xi_1) \cap BGRP(\xi_2)$.

Thus, by (7) we have Proposition 88.

Proposition 89.

$\psi(b_2) = (\pi_{i_2}(j_2), \rho_{j_2}(i_2))$ for each $b_2 = (i_2, j_2) \in B_2$.

Proof.

We can define a map $\psi^* : B_2 \to B_1$ by

(1) $\psi^*(b_2) = (\pi_{i_2}(j_2), \rho_{j_2}(i_2))$ for each $b_2 = (i_2, j_2) \in B_2$.

Take any $b_2 = (i_2, j_2) \in B_2$, we have that

(2) $BEXP(b_2, \psi^*(b_2))$

$= BEXP((i_2, j_2), (\pi_{i_2}(j_2), \rho_{j_2}(i_2)))$

$= (EXP(i_2, \pi_{i_2}(j_2)), EXP(j_2, \rho_{j_2}(i_2)))$

$= (\xi_1(b_2), \xi_2(b_2))$.

By (3) in the proof of Proposition 88, we have

(3) $BEXP(b_2, \psi(b_2)) = (\xi_1(b_2), \xi_2(b_2))$.

By (2), (3) we have that

(4) $BEXP(b_2, \psi^*(b_2)) = BEXP(b_2, \psi(b_2))$.

By the uniquness of Proposition 82 we have that

(4) $\psi = \psi^*$.

Hence, we have Proposition 89.

By our observations in this section we have the following:

Proposition 90.

Let $T \subset J_1 \times J_2$ be a basic set. Then we have the followings:

(i) unique bijection maps $\pi_{i_2} : Z_{(2,2)} \to Z_{(1,1)}$ for $i_2 \in Z_{(1,2)}$

(ii) unique bijective maps $\rho_{j_2} : Z_{(1,2)} \to Z_{(2,1)}$ for $j_2 \in Z_{(2,2)}$

(iii) the unique map $\psi : B_2 \to B_1$, defined by

$\psi(b_2) = (\pi_{i_2}(j_2), \rho_{j_2}(i_2))$ for each $b_2 = (i_2, j_2) \in B_2$,

such that

(iv) $T \cap I(b_2) = \{BEXP(b_2, \psi(b_2))\}$ for each $b_2 \in B_2$.

Moreover, we have

(v) maps $\xi_1 : B_2 \to J_1$ and $\xi_2 : B_2 \to J_2$ defined by

$\xi_1(b_2) = EXP(i_2, \pi_{i_2}(j_2))$ for $b_2 = (i_2, j_2) \in B_2$,

$\xi_2(b_2) = EXP(j_2, \rho_{j_2}(i_2))$ for $b_2 = (i_2, j_2) \in B_2$,

such that

(vi) $T = BGRP(\xi_1) \cap BGRP(\xi_2)$.

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