Mathematics and Suduku VII

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We discuss on the worldwide famous Sudoku by using mathematical approach. This paper is the 7th paper in our series, so we use the same notations and terminologies in [1] without any descriptions.

11. Latin squares and coordinate transformations.

A map $f:J_1\times J_2\to J_3$ is a Latin square map provieded that it satisfies the following condition:

(LSM) $f \mid b:b\to J_3$ is bijective for each $b\in rOW\cup cOL$.
Here rOW is the set of all rows and cOL is the set of all columns of $J_1\times J_2$, respectively.

Let L_0 be a subset of $J_1\times J_2$ and $f_0:L_0\to J_3$ be a map. A map $f:J_1\times J_2\to J_3$ is a Latin square solution map of f_0 provided that it satisfies the following condition:

(SOL) f is a Latain square map with $f\mid L_0=f_0$.

Proposition 56.
Let $f:J_1\times J_2\to J_3$ be a Latin square map. Then we have the unique maps $g:J_3\times J_2\to J_1$ and $h:J_1\times J_3\to J_2$ with the following conditions:
(i) $f(g(\beta),j)=k$ for each $\beta=(k,j)\in J_3\times J_2$.
(ii) $f(i,h(\gamma))=k$ for each $\gamma=(i,k)\in J_1\times J_3$.
Proof. Take any $\beta=(k,j)\in J_3\times J_2$. We put $c_j=\{(i,j) : i\in J_1\}\in cOL$. Since $c_j\subset J_1\times J_2$, by (LSM) we have that
(1) $f\mid c_j:c_j\to J_3$ is bijective.
By (1), there exists the unique $i_0\in J_1$ such that
(2) $(i_0,j)\in c_j$ and
(3) $f(i_0,j)=k$.
Thus, we can define a map $g:J_3\times J_2\to J_1$ by $g(\beta)=i_0$.
Therefore, by (3) we have that
(4) $f(g(\beta),j)=k$
(4) means the condition (i).

Next, we consider the uniqueness of g.
Let $g, g':J_3\times J_2\to J_1$ be maps with the condition (i).
Thus, we have that
(5) $f(g(\beta),j)=k$ for each $\beta=(k,j)\in J_3\times J_2$,
(6) $f(g'(\beta),j)=k$ for each $\beta=(k,j)\in J_3\times J_2$.

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By (1), (5) and (6) we have that
(7) $g(\beta)=g'(\beta)$.
Therefore, we have that $g=g'$. Hence we have the uniquness of g.

By similar ways we can show (ii). Hence we have Proposition 56.

The maps g and h in Proposition 56 are induced by f, thus, we put $g=\omega_{(1,3)}(f)$ and $h=\omega_{(2,3)}(f)$, respectively.

Proposition 57.
Let $f:J_1\times J_2\to J_3$ be a Latin square map. Then $g=\omega_{(1,3)}(f):J_3\times J_2\to J_1$ and $h=\omega_{(2,3)}(f):J_1\times J_3\to J_2$ are also Latain square maps.
Proof. We show the following Claims:

Claim 1.
$g\mid c_j:c_j\to J_1$ is bijective for each $j\in J_2$ and $c_j=\{(k,j):k=1,2,...,9\}\subset J_3\times J_2$.

Claim 2.
$g\mid r_k:r_k\to J_1$ is bijective for each $k\in J_3$ and $r_k=\{(k,j):j=1,2,...,9\}\subset J_3\times J_2$.

Proof of Claim 1.
Take any $j\in J_2$. First, we show that $g\mid c_j:c_j\to J_1$ is surjective.
Take any $i\in J_1$. We put
(1) $k=f(i,j)\in J_3$,
(2) $\beta=(k,j)\in c_j\subset J_3\times J_1$.
By (i) of Proposition 56
(3) $f(g(\beta),j)=k$.
By (1), (3) we have
(4) $f(i,j)=f(g(\beta),j)$.
By (LSM) of f for columns and (4) we have that
(5) $g(\beta)=i$.
Thus, by (5), $g\mid c_j:c_j\to J_1$ is surjective.

Next, take any $j\in J_2$. We show that $g\mid c_j:c_j\to J_1$ is injective.
Take any $\beta_1=(k_1,j)$, $\beta_2=(k_2,j)\in c_j$ such that
(6) $g(\beta_1)=g(\beta_2)$.
By (i) of Proposition 56 we have
(7) $f(g(\beta_1),j)=k_1$,

(8) $f(g(\beta_2),j)=k_2$.
By (6),(7),(8) we have
(9) $k_1=k_2$.
By (9) we have
(10) $\beta_1=\beta_2$.
Hence, by (10) $g \mid c_j : c_j \to J_1$ is injective.
Therefore, we have Claim 1.

Proof of Claim 2.
Take any $k \in J_3$. We show that $g \mid r_k : r_k \to J_1$ is surjective.
Take any $i \in J_1$. By (LSM) of f for rows, there exists a $j_0 \in J_2$ such that
(11) $f(i,j_0)=k$.
We put
(12) $\beta_3=(k,j_0) \in r_k \subset J_3 \times J_2$.
By (i) of Proposition 56, we have that
(13) $f(g(\beta_3),j_0)=k$.
By (11), (13) and (LSM) of f for columns, we have that
(14) $g(\beta_3)=i$.
By (14) we have that $g \mid r_k : r_k \to J_1$ is surjective.

Next, take any $k \in J_3$.
we show that $g \mid r_k : r_k \to J_1$ is injective.
Take any $\beta_4=(k,j_4)$, $\beta_5=(k,j_5) \in r_k \subset J_3 \times J_2$ such that
(15) $g(\beta_4)=g(\beta_5)=i_1$.
By (i) of Proposition 56, we have that
(16) $f(g(\beta_4),j_4)=k$,
(17) $f(g(\beta_5),j_5)=k$.
By (15), (16), (17) we have that
(18) $f(i_1,j_4)=k$,
(19) $f(i_1,j_5)=k$.
By (LSM) of f for rows, and (18), (19) we have that
(20) $j_4=j_5$.
By (20) we have that $\beta_4=(k,j_4)=(k,j_5)=\beta_5$, i.e.,
(21) $\beta_4=\beta_5$.
Thus, by (21) we have that $g \mid r_k : r_k \to J_1$ is injective.
Therefore, we have Claim 2.

By Claim 1 and Claim 2, $g=\omega_{(1,3)}(f): J_3 \times J_2 \to J_1$ forms a Latin square map.

By similar ways, we can show that $h=\omega_{(2,3)}(f): J_1 \times J_3 \to J_2$ forms a Latin square map.
Hence, we have Proposition 57.

By Proposition 57, $g=\omega_{(1,3)}(f): J_3 \times J_2 \to J_1$ forms a Latin square map. Thus also we have the Latin square map $\omega_{(1,3)}(g): J_1 \times J_2 \to J_3$.
In these Latin square maps we have the following Proposition.

Proposition 58.
Let $f: J_1 \times J_2 \to J_3$ be a Latin square map. Then we have that
(i) $\omega_{(1,3)}(\omega_{(1,3)}(f))=f$,
(ii) $\omega_{(2,3)}(\omega_{(2,3)}(f))=f$.
Proof. We show (i).
We put $g=\omega_{(1,3)}(f)$ and $f^*=\omega_{(1,3)}(g)$. Thus by (i) of Proposition 56, we have that
(1) $f(g(\beta),j)=k$ for each $\beta=(k,j) \in J_3 \times J_2$,
(2) $g(f^*(\alpha),j)=i$ for each $\alpha=(i,j) \in J_1 \times J_2$.
Take any $\alpha=(i,j) \in J_1 \times J_2$. We put
(3) $f(i,j)=f(\alpha)=k$.
We put $\beta=(k,j)$. Thus by (1) we have that
(4) $f(g(\beta),j)=k$.
By (LSM) for the column c_j ,(3),(4) we have that
(5) $g(k,j)=g(\beta)=i$.
By Proposition 57, g is a Latin square map. Thus, g satisfies the condition (LSM). Then by (2), (5) and (LSM) for g, we have that
(6) $f^*(\alpha)=k$.
By (3),(6) we have that
(7) $f(\alpha)=f^*(\alpha)=k$.
By (7) we have that
(8) $f=f^*$.
Hence, by (8) we have (i).
By similar ways, we can show (ii).
Therefore, we have Proposition 58.

Let $f: J_1 \times J_2 \to J_3$ be a Latin square map. Nowwe make a new map $f^t: J_2 \times J_1 \to J_3$ as follows:
$f^t(j,i)=f(i,j)$ for each $(j,i) \in J_2 \times J_1$.
And we put $f^t=\omega_{(1,2)}(f)$.

Proposition 59.
Let $f: J_1 \times J_2 \to J_3$ be a Latin square map. We have the followings:
(i) $f^t=\omega_{(1,2)}(f): J_2 \times J_1 \to J_3$ is also a Latin square map.
(ii) $\omega_{(1,2)}(\omega_{(1,2)}(f))=f$.
(iii) $\omega_{(1,3)}(f)=\omega_{(1,2)}(\omega_{(2,3)}(\omega_{(1,2)}(f)))$.
(iv) $\omega_{(2,3)}(f)=\omega_{(1,2)}(\omega_{(1,3)}(\omega_{(1,2)}(f)))$.
Proof. We can easily show (i) and (ii) by using the definition of f^t.

We show (iii). Let $h_1=\omega_{(2,3)}(f^t): J_2 \times J_3 \to J_1$ and $(h_1)^t=\omega_{(1,2)}(h_1): J_3 \times J_2 \to J_1$. By Proposition 56, we have
(1) $f^t(j,h_1(j,k))=k$ for each $(j,k) \in J_2 \times J_3$.
By (1) we have that for each $(k,j) \in J_3 \times J_2$,

$f((h_1)^t(k,j),j)=f^t(j,(h_1)^t(k,j))=f^t(j,h_1(j,k))=k$, i.e.,

(2) $f((h_1)^t(k,j),j)=k$ for each $(k,j)\in J_3\times J_2$.

Let $g=\omega_{(1,3)}(f):J_3\times J_2\to J_1$. By Proposition 56 we have that

(3) $f(g(k,j),j)=k$ for each $(k,j)\in J_3\times J_2$.

By (2), (3) and the uniquenss of Proposition 56, we have that

(4) $g=(h_1)^t$.

Thus, by (4), we have

$\omega_{(1,3)}(f)=g=(h_1)^t=\omega_{(1,2)}(h_1)=\omega_{(1,2)}(\omega_{(2,3)}(f^t))$

$=\omega_{(1,2)}(\omega_{(2,3)}(\omega_{(1,2)}(f)))$, i.e.,

(5) $\omega_{(1,3)}(f)=\omega_{(1,2)}(\omega_{(2,3)}(\omega_{(1,2)}(f)))$.

By (5) we have (iii).

We show (iv).

Since we have (iii), we use (iii) for $\omega_{(1,2)}(f)$. Thus we have

(6) $\omega_{(1,3)}(\omega_{(1,2)}(f))=\omega_{(1,2)}(\omega_{(2,3)}(\omega_{(1,2)}(\omega_{(1,2)}(f))))$.

By (ii) and (6) we have that

(7) $\omega_{(1,3)}(\omega_{(1,2)}(f))=\omega_{(1,2)}(\omega_{(2,3)}(f))$.

By (7) we have that

(8) $\omega_{(1,2)}(\omega_{(1,3)}(\omega_{(1,2)}(f)))=\omega_{(1,2)}(\omega_{(1,2)}(\omega_{(2,3)}(f)))$.

By (ii) and (8) we have that

(9) $\omega_{(1,2)}(\omega_{(1,3)}(\omega_{(1,2)}(f)))=\omega_{(2,3)}(f)$.

By (9) we have (iv).

Therefore, we have Proposition 59.

Remark 60.

(a) Every sudoku map is a Latinsquare map. However, we have many Latin square maps which are not sudoku maps.

(b) If f is a sudoku map, then $\omega_{(1,2)}(f)$ is also a sudoku map.

(c) There are many sudoku maps f, but $\omega_{(1,3)}(f)$ and $\omega_{(2,3)}(f)$ are not sudoku maps.

12. Sudoku matrices and coordinate trans-formations.

Let $\sigma:\{1,2,3\}\to\{1,2,3\}$ be a bijective map. Usually, we say that σ is a permutation on theset $\{1,2,3\}$. Also σ is denoted as $\sigma=\begin{pmatrix}1,2,3\\ \sigma(1),\sigma(2),\sigma(3)\end{pmatrix}$.

Let $\tau:\{1,2,3\}\to\{1,2,3\}$ be an another bijective map. Then $\tau\sigma$ is defined by $(\tau\sigma)(i)=\tau(\sigma(i))$ for each $i\in\{1,2,3\}$. Then $\tau\sigma$ is also bijective.

For each permutation $\sigma=\begin{pmatrix}1,2,3\\ \sigma(1),\sigma(2),\sigma(3)\end{pmatrix}$ of the set $\{1,2,3\}$. we define a transformation

$T_\sigma:J_1\times J_2\times J_3\to J_{\sigma(1)}\times J_{\sigma(2)}\times J_{\sigma(3)}$ by

$T_\sigma(x_1,x_2,x_3)=(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$

for any $(x_1,x_2,x_3)\in J_1\times J_2\times J_3$.

Special cases, usually, we use the notations as follows:

$\begin{pmatrix}1,2,3\\3,2,1\end{pmatrix}=(1,3)$, $\begin{pmatrix}1,2,3\\1,3,2\end{pmatrix}=(2,3)$, $\begin{pmatrix}1,2,3\\2,1,3\end{pmatrix}=(1,2)$

and so on.

Thus we can also use the notations :

$T_{(1,2)}:J_1\times J_2\times J_3\to J_2\times J_1\times J_3$

$T_{(1,3)}:J_1\times J_2\times J_3\to J_3\times J_2\times J_1$

$T_{(2,3)}:J_1\times J_2\times J_3\to J_1\times J_3\times J_2$

are defined as follows:

$T_{(1,2)}(i,j,k)=(j,i,k)$

$T_{(1,3)}(i,j,k)=(k,j,i)$

$T_{(2,3)}(i,j,k)=(i,k,j)$

for each $(i,j,k)\in J_1\times J_2\times J_3$.

We can easily show Proposition 61.

Proposition 61.

We have the following:

(i) $T_\tau T_\sigma=T_{\tau\sigma}$ for each permutations σ and τ.

(ii) $T_{\sigma_0}=1$ for $\sigma_0=\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}=1$.

For a map $\psi:X\to Y$ we define the graph $GRH(\psi)$ of ψ as follows:

$GRH(\psi)=\{(x,\psi(x)):x\in X\}\subset X\times Y$.

For Latin square maps f,g,h in Proposition 56 and f^t in Proposition 59 we have graphs as follows:

$GRH(f)=\{(\alpha,f(\alpha)):\alpha\in J_1\times J_2\}\subset J_1\times J_2\times J_3$,

$GRH(g)=\{(\beta,g(\beta)):\beta\in J_3\times J_2\}\subset J_3\times J_2\times J_1$,

$GRH(h)=\{(\gamma,h(\gamma)):\gamma\in J_1\times J_3\}\subset J_1\times J_3\times J_2$,

$GRH(f^t)=\{(\delta,f^t(\delta)):\delta\in J_2\times J_1\}\subset J_2\times J_1\times J_3$.

Proposition 62.

Let $f:J_1\times J_2\to J_3$ be a Latin square map.

(a) For each $\sigma\in\{(1,2),(1,3),(2,3)\}$,

$T_\sigma:J_1\times J_2\times J_3\to J_{\sigma(1)}\times J_{\sigma(2)}\times J_{\sigma(3)}$

induces an map

$t_\sigma^f=T_\sigma\mid GRP(f):GRP(f)\to GRP(\omega_\sigma(f))$.

(b) For each $\sigma\in\{(1,2),(1,3),(2,3)\}$,

$t_\sigma^f:GRP(f)\to GRP(\omega_\sigma(f))$ and $t_\sigma^{\omega_\sigma(f)}:GRP(\omega_\sigma(f))\to GRP(f)$ satisfy that

$t_\sigma^{\omega_\sigma(f)}\circ t_\sigma^f=1$ and $t_\sigma^f\circ t_\sigma^{\omega_\sigma(f)}=1$.

(c) t_σ^f is bijective.

Proof. Take any $(\alpha,f(\alpha))\in GRP(f)$. We put

(1) $\alpha=(i_0,j_0)\in J_1\times J_2$ and

(2) $k_0=f(\alpha)=f(i_0,j_0)\in J_3$.

Claim 1.
(a) holds for $\sigma=(1,3)$.
By definitions, we have that
(3) $T_{(1,3)}(\alpha,f(\alpha))=T_{(1,3)}(i_0,j_0,k_0)=(k_0,j_0,i_0)$,
(4) $g=\omega_{(1,3)}(f):J_3\times J_2\to J_1$.
By Proposition 58 we have
(5) $f=\omega_{(1,3)}(g):J_1\times J_2\to J_3$.
By Proposition 56 for g and (5) we have that
(6) $g(f(i,j),j)=i$ for each $(i,j)\in J_1\times J_2$.
By (2), (6) we have that
(7) $g(k_0,j_0)=g(f(i_0,j_0),j_0)=i_0$.
Thus, we put $\beta=(k_0,j_0)\in J_3\times J_2$ and by (7)
(8) $g(\beta)=i_0$.
By (8) we have that
(9) $(k_0,j_0,i_0)=(\beta,g(\beta))\in GRP(g)$.
By (3), (9) we have that
(10) $T_{(1,3)}(\alpha,f(\alpha))=(\beta,g(\beta))\in GRP(g)$.
Thus, by (10), we have that
(11) $T_{(1,3)}(GRP(f))\subset GRP(g)$.
By (11), $T_{(1,3)}$ induces a map
(12) $t^f_{(1,3)}:GRP(f)\to GRP(g)$.
Hence, we have Claim 1.

Claim 2.
(b) and (c) hold for $\sigma=(1,3)$.
We can apply Claim 1 for the Latin square map $g=\omega_{(1,3)}(f)$. That is,
$T_\sigma(GRH(g))\subset GRH(\omega_\sigma(g))$.
Thus T_σ induces a map
(13) $t^g_{(1,3)}:GRH(g)\to GRH(\omega_{(1,3)}(g))$.
By Proposition 58 and (4), we have that
(14) $\omega_{(1,3)}(g)=\omega_{(1,3)}(\omega_{(1,3)}(f))=f$.
Then, by (13), (14) we have that
(15) $t^g_{(1,3)}:GRP(g)\to GRP(f)$.
By Proposition 61, we have that
(16) $T_{(1,3)}T_{(1,3)}=T_{(1,3)(1,3)}=T_{\sigma_0}=1$.

Since the maps $t^f_{(1,3)}$ and $t^g_{(1,3)}$ are induced by $T_{(1,3)}$, by (12) and (13), (16) we have that
(17) $t^g_{(1,3)}\circ t^f_{(1,3)}=1$ and $t^f_{(1,3)}\circ t^g_{(1,3)}=1$.
By (17) we can easily show that
(18) $t^f_{(1,3)}$ and $t^g_{(1,3)}$ are bijective.
Thus, by (17), (18) we have Claim 2.

By Claim 1 and Claim 2, we have (a), (b), (c) for $\sigma=(1,3)$.

By similar ways we can easily show (a), (b), (c)for other $\sigma\in\{(1,2),(2,3)\}$. Hence we have Proposition 62.

Remark 63.
We put $\sigma_0=\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}=1$, $\omega_{\sigma_0}(f)=f$ and $S=\{\sigma_0,(1,2),(1,3),(2,3)\}$. By Proposition 62, we have maps $t^{\omega_\sigma(f)}_\sigma:GRP(\omega_\sigma(f))\to GRP(f)$ and $t^f_\tau:GRP(f)\to GRP(\omega_\tau(f))$ for each $\sigma,\tau\in S$. Then we have $t^f_\tau\circ t^{\omega_\sigma(f)}_\sigma:GRP(\omega_\sigma(f))\to GRP(\omega_\tau(f))$ which is induced by $T_\tau\circ T_\sigma=T_{\tau\sigma}$. Hence, $t^f_\tau\circ t^{\omega_\sigma(f)}_\sigma$ is bijective.

Let $f:J_1\times J_2\to J_3$ is a Latin square map. Let $K=(K_\alpha)_{\alpha\in J_1\times J_2}$ be a Latin square matrix associated with f provided that it satisfies the conditions:
(LMTX) $f(\alpha)\in K_\alpha\subset J_3$ for each $\alpha\in J_1\times J_2$.
Let $LMTX(f)$ be the set of all Latin square matrices K associated with f.

A set $U\subset J_1\times J_2\times J_3$ is a neighborhood of f provided that it satisfies the condition
(NBH) $GRP(f)\subset U$.
Let $NBH(f)=\{U: U\ is\ a\ neighborhood\ of\ f\}$.

We define maps $\eta_f:LMTX(f)\to NBH(f)$ and $\theta_f:NBH(f)\to LMTX(f)$ as follows:
For each $K=(K_\alpha)_{\alpha\in J_1\times J_2}\in LMTX(f)$, we put $\eta_f(K)=U(K)=\{(i,j,k)\in J_1\times J_2\times J_3:k\in K_{(i,j)}\}$.

For each $U\in NBH(f)$, we put $\theta_f(U)=K(U)$, $K(U)=(U_\alpha)_{\alpha\in J_1\times J_2}$ and $U_\alpha=\{k: (i,j,k)\in U\}$ for each $\alpha=(i,j)\in J_1\times J_2$.

We easily show that the maps η_f and θ_f are well-defined. And we can easily show the following:

Proposition 64.
Let $K,L\in LMTX(f)$ and $U,V\in NBH(f)$. We have the followings:
(a) $\eta_f:LMTX(f)\to NBH(f)$ and $\theta_f:NBH(f)\to LMTX(f)$ are well-defined.
(b) $\theta_f\circ\eta_f=1$ and $\eta_f\circ\theta_f=1$.
(i) θ_f and η_f are bijective.
(c) If $L\leqq K$, then $\eta_f(L)\subset\eta_f(K)$.
(d) If $V\subset U$, then $\theta_f(V)\leqq\theta_f(U)$.

In the proof of Proposition 62 we show that for each $\sigma\in\{\sigma_0, (1,2),(1,3),(2,3)\}$,
$T_\sigma:J_1\times J_2\times J_3\to J_{\sigma(1)}\times J_{\sigma(2)}\times J_{\sigma(3)}$
induces
$T_\sigma(GRP(f))=GRP(\omega_\sigma(f))$.

Hence, we can define
$T^*_\sigma(f): NBH(f) \to NBH(\omega_\sigma(f))$ by
$T^*_\sigma(f)(U) = T_\sigma(U)$ for each $U \in NBH(f)$.

Since $T_\sigma T_\sigma = 1$, we can easily show the following:

Proposition 65.
Let $\sigma \in \{\sigma_0, (1,2), (1,3), (2,3)\}$ and $U, V \in NBH(f)$.
(a) $T^*_\sigma(f): NBH(f) \to NBH(\omega_\sigma(f))$ is well-defined.
(b) $T^*_\sigma(\omega_\sigma(f)) \circ T^*_\sigma(f) = 1$, $T^*_\sigma(f) \circ T^*_\sigma(\omega_\sigma(f)) = 1$.
(i) $T^*_\sigma(f)$, $T^*_\sigma(\omega(f))$ are bijective.
(c) If $V \subset U$, then $T^*_\sigma(f)(V) \subset T^*_\sigma(f)(U)$.

Now, for each $\sigma \in \{\sigma_0, (1,2), (1,3), (2,3)\}$, we can define $MX_\sigma(f): LMTX(f) \to LMTX(\omega_\sigma(f))$ by
$MX_\sigma(f) = \theta_{\omega_\sigma(f)} \circ T^*_\sigma(f) \circ \eta_f$:
$LMTX(f) \to NBH(f) \to NBH(\omega_\sigma(f)) \to LMTX(\omega_\sigma(f))$
By Proposition 64 and Proposition 65 we have the following.

Proposition 66.
For each $\sigma \in \{\sigma_0, (1,2), (1,3), (2,3)\}$, each Latin square map $f: J_1 \times J_2 \to J_3$ induces the bijective map
$MX_\sigma(f): LMTX(f) \to LMTX(\omega_\sigma(f))$
and it satisfies the followings:
(a) $MX_\sigma(\omega_\sigma(f)) \circ MX_\sigma(f) = 1$,
$MX_\sigma(f) \circ MX_\sigma(\omega_\omega(f)) = 1$.
(b) Let $K, K' \in LTMX(f)$. If $K' \leqq K$, then $MX_\sigma(f)(K') \leqq MX_\sigma(f)(K)$.

We recall intersectable n-systems, which are called by n-igeta systems.

We recall the notations as follows:
For each $i \in J_1$, $r_i = \{(i,j): j \in J_2\} \subset J_1 \times J_2$ is the $i-th$ row and for each $j \in J_2$, $c_j = \{(i,j): i \in J_1\} \subset J_1 \times J_2$ is the $j-th$ column.

Let n be an integer, $1 \leqq n \leqq 9$.
Let $A = \{i_1, i_2, \ldots, i_n\} \subset J_1$, $B = \{j_1, j_2, \ldots, j_n\} \subset J_2$, $k_0 \in J_3$,
$R(A) = \cup\{r_i : i \in A\} = A \times J_2$,
$C(B) = \cup\{c_j : j \in B\} = J_1 \times B$.
Thus, $R(A) \cap C(B) = A \times B$.

Let $K = (K_\alpha)_{\alpha \in J_1 \times J_2} \in LMTX(f)$.
We consider the following conditions for K:
(CI) $k_0 \notin K_\alpha$ for each $\alpha \in C(B) - R(A)$.
(RI) $k_0 \notin K_\alpha$ for each $\alpha \in R(A) - C(B)$.
(RC) $K^* = (K^*_\alpha)_{\alpha \in J_1 \times J_2} \in LMTX(f)$. Here

$$K^*_\alpha = \begin{cases} K_\alpha - \{k_0\} & for\ \alpha \in R(A) - C(B) \\ K_\alpha & for\ \alpha \in J_1 \times J_2 - (R(A) - C(B)) \end{cases}$$

(CR) $K^* = (K^*_\alpha)_{\alpha \in J_1 \times J_2} \in LMTX(f)$. Here

$$K^*_\alpha = \begin{cases} K_\alpha - \{k_0\} & for\ \alpha \in C(B) - R(A) \\ K_\alpha & for\ \alpha \in J_1 \times J_2 - (C(B) - R(A)) \end{cases}$$

Let $i_0 \in J_1$, $j_0 \in J_2$. We consider the following conditions for K:
(CNSF) $|K(A \times \{j_0\})| = |A|$
(RNSF) $|K(\{i_0\} \times B)| = |B|$.
Here, $K(S) = \cup\{K_\alpha : \alpha \in S\}$ for each $S \subset J_1 \times J_2$.

Proposition 67.
Let $K = (K_\alpha)_{\alpha \in J_1 \times J_2} \in LMTX(f)$. We have that
(a) (CI) implies (RC).
(b) (RI) implies (CR).
Proof. We show (a).

Claim 1.
$f(\alpha) \neq k_0$ for each $\alpha \in C(B) - R(A)$.
Since $K \in LMTX(f)$,
(1) $f(\alpha) \in K_\alpha$ for each $\alpha \in J_1 \times J_2$.
By (CI) we have that
(2) $k_0 \notin K_\alpha$ for each $\alpha \in C(B) - R(A)$.
By (1),(2) we have that
(3) $f(\alpha) \neq k_0$ for each $\alpha \in C(B) - R(A)$.
Hence, we have Claim 1.

Let $g = \omega_{(1,3)}(f): J_3 \times J_2 \to J_1$. We define a map $g_{k_0}: J_2 \to J_1$ by
(4) $g_{k_0}(j) = g(k_0, j)$ for each $j \in J_2$.

Claim 2.
$g_{k_0}(B) \subset A$.
We assume that Claim 2 does not hold. Thus, we have that
(5) $j_0 \in B$
(6) $g(k_0, j_0) \notin A$.
We put
(7) $i_0 = g(k_0, j_0) \notin A$.
By definition of g, we have that
(8) $f(g(k,j),j) = k$ for each $(k,j) \in J_3 \times J_2$.
Then we have
(9) $f(g(k_0,j),j) = k_0$ for each $j \in J_2$.
Thus, by (9) we have
(10) $f(g(k_0,j_0),j_0) = k_0$.
By (7),(10) we have
(11) $f(i_0,j_0) = k_0$.

By (5),(7) we have
(12) $(i_0,j_0) \in C(B)-R(A)$.
(11) and (12) contradict to Claim 1.
Hence we have Claim 2.

Claim 3.
$g_{k_0}(B)=A$.
By Proposition 57, g is a Latin square map. Thus we have that
(13) g_{k_0} is bijective.
By (13) we have that
(14) $n = |B| = |g_{k_0}(B)|$
By Claim 2 we have that
(15) $|g_{k_0}(B)| \leqq |A| = n$
By (14), (15) and Claim 2 we have that
(16) $g_{k_0}(B)=A$.
By (16) we have Claim 3.

Claim 4.
$f(\beta) \neq k_0$ for each $\beta \in R(A)-C(B)$.
We assume that Claim 5 does not hold. Thus, we have some β_1 such that
(17) $\beta_1=(i_1,j_1) \in R(A)-C(B)$, and
(18) $f(i_1,j_1)=f(\beta_1)=k_0$.
By (17) we have
(19) $i_1 \in A$ and $j_1 \notin B$.
By (19), Claim 3 we have
(20) $j_2 \in B$ and
(21) $i_1=g_{k_0}(j_2)=g(k_0,j_2)$.
By (8) we have that
(22) $f(g(k_0,j_2),j_2)=k_0$.
By (21), (22) we have that
(23) $f(i_1,j_2)=f(g(k_0,j_2),j_2)=k_0$.
By (18), (23) we have
(24) $f(i_1,j_1)=f(i_1,j_2)=k_0$.
Since f is a Latin square map,
(25) $f \mid r_{i_1}: r_{i_1} \to J_3$ is bijective.
Here, $r_{i_1}=\{(i_1,j): j \in J_2\}$ is the i_1-th row. By (24), (25) we have that
(26) $j_1=j_2$.
By (19),(20),(26)
(27) $j_1 \notin B \ni j_2=j_1$.
By (27) we have a contradiction.
Hence, we have Claim 4.

Since $K \in LMTX(f)$, we have that
(28) $f(\alpha) \in K_\alpha$ for each $\alpha \in J_1 \times J_2$.
By Claim 4, (28) we have that
(29) $K_\beta - \{k_0\} \ni f(\beta)$ for each $\beta \in R(A)-C(B)$.

By (29) we can easily show (RC).
Hence, we have (a).

By similar ways we can show (b). Thus we have Proposition 67.

Proposition 68.
Let $f: J_1 \times J_2 \to J_3$ be a Latin square map and $g=\omega_{(1,3)}(f)$. Let $K \in LMTX(f)$ and $L_{(1,3)}=MX_{(1,3)}(f)(K) \in LMTX(g)$. Let $A \subset J_1$, $B \subset J_2$, $|A| = |B| = n$ and $k_0 \in J_3$.
Then the following (a) and (b) are equivalent:
(a) K satisfies the condition (CI) :
(CI) $k_0 \notin K_\alpha$ for each $\alpha \in C(B)-R(A)$.
(b) $L_{(1,3)}$ satisfies the conditin (RNSF):
(RNSF) $L_{(1,3)}(\{k_0\} \times B)=A$.
Proof. We put $K=(K_\alpha)_{\alpha \in J_1 \times J_2}$, $L_{(1,3)}=(L_\beta)_{\beta \in J_3 \times J_2}$, $A=\{i_1,i_2,\dots,i_n\} \subset J_1$ and $B=\{j_1,j_2,\dots,j_n\} \subset J_2$.

We show that (a)→(b).
By definitions we can show that
(1) $L_\beta=\{i \in J_1: k \in K_{(i,j)}\}$ for each $\beta=(k,j) \in J_3 \times J_2$.
Let $S=\{(k_0,j): j \in B\}=\{k_0\} \times B \subset \{k_0\} \times J_2$.

Claim 1.
$L_\beta \subset A$ for each $\beta \in S$.
Take any $\beta \in S$ and put $\beta=(k_0,j_s), j_s \in B$. Take any $i \in L_\beta$. By (1) we have that
(2) $k_0 \in K_{\alpha_0}$ and $\alpha_0=(i,j_s)$.
We show that
(3) $i \in A$.

We assume that
(4) $i \notin A$.
By (4) and $j_s \in B$, we have that
(5) $\alpha_0=(i,j_s) \in C(B)-R(A)$.
By (a) and (5), we have that
(6) $k_0 \notin K_{\alpha_0}$.
Since (6) contradicts to (2), we have (3) i.e., $L_\beta \subset A$.
Hence, we have Claim 1.

Claim 2.
$\cup\{L_\beta: \beta \in S\}=A$.
By Claim 1 we have that
(7) $\cup\{L_\beta: \beta \in S\} \subset A$.
Since $L_{(1,3)} \in LMTX(g)$, we have that
(8) $g(\beta) \in L_\beta$ for each $\beta \in J_3 \times J_2$.
Then we have that
(9) $g(S) \subset \cup\{L_\beta: \beta \in S\}$.
Thus, by (7), (9) we have that

(10) $g(S)\subset \cup\{L_\beta:\beta\in S\}\subset A$

Since $g:J_3\times J_2\to J_1$ is a Latin square map by Proposition 57, we have that

(11) $g\mid\{k_0\}\times J_2:\{k_0\}\times J_2\to J_1$ is bijective.

Since $S=\{(k_0,j):j\in B\}=\{k_0\}\times B\subset\{k_0\}\times J_2$, by (11)

(12) $g\mid S:S\to J_1$ is injective.

By (12) we have that

(13) $|g(S)|=|S|=n$.

By definitions, we have that

(14) $|S|=|B|=|A|=n$.

By (10), (13), (14) we have that

(15) $g(S)=\cup\{L_\beta:\beta\in S\}=A$.

Hence, by (15) we have Claim 2.

Since $L_{(1,3)}(\{k_0\}\times B)=\cup\{L_\beta:\beta\in S\}$, by Claim 2 we have

(16) $L_{(1,3)}(\{k_0\}\times B)=A$.

By (16) we have (b).

We show that (b)→(a).

By Proposition 57

(17) $g=\omega_{(1,3)}(f)$. $g:J_3\times J_2\to J_1$ is aLatin square map.

By (b), $L_{(1,3)}\in \boldsymbol{LMTX}(g)$ has

(18) $A=L_{(1,3)}(\{k_0\}\times B)=\cup\{L_\beta:\beta\in\{k_0\}\times B\}$.

By (17) we can put $M_{(1,3)}=\boldsymbol{MX}_{(1,3)}(g)(L_{(1,3)})$ and also put $M_{(1,3)}=(M_\alpha)_{\alpha\in J_1\times J_2}\in \boldsymbol{LMTX}(\omega_{(1,3)}(g))$.

By the definitions we have that

(19) $M_\alpha=\{k\in J_3:\ i\in L_{(k,j)}\}$

for each $\alpha=(i,j)\in J_1\times J_2$.

Claim 3.

$k_0\notin M_\alpha$ for each $\alpha\in C(B)-R(A)$.

We assume that Claim 3 does not hold. Then, there is an $\alpha_0\in C(B)-R(A)$ such that

(20) $k_0\in M_{\alpha_0}$, $\alpha_0\in C(B)-R(A)$.

We put $\alpha_0=(i_0,j_0)$. Since $\alpha_0\in C(B)-R(A)$, then

(21) $i_0\notin A$ and $j_0\in B$.

By (19), (20) we have that

(22) $i_0\in L_{(k_0,j_0)}$.

Since $j_0\in B$ by (21),

(23) $\beta_0=(k_0,j_0)\in\{k_0\}\times B$.

By (18), (22), (23) we have that

(24) $i_0\in L_{\beta_0}\subset\cup\{L_\beta:\beta\in\{k_0\}\times B\}=A$.

Thus, (24) contradicts to (21).

Hence, we have Claim 3.

By Proposition 58 and Proposition 66, we have that

(25) $f=\omega_{(1,3)}(g)$ and $K=M_{(1,3)}$.

By (25) and Claim 3 we have that

Claim 4.

$k_0\notin K_\alpha$ for each $\alpha\in C(B)-R(A)$.

By Claim 4 we have (a).

Therefore, (a) and (b) are equivalent. Hence we have Proposition 68.

By the similar way as Proposition 68 we can show the following Proposition 69.

Proposition 69.

Let $f:J_1\times J_2\to J_3$ be a Latin square map and $h=\omega_{(2,3)}(f)$. Let $K\in \boldsymbol{LMTX}(f)$ and $L_{(2,3)}=\boldsymbol{MX}_{(2,3)}(f)(K)\in \boldsymbol{LMTX}(h)$. Let $A\subset J_1$, $B\subset J_2$, $|A|=|B|=n$ and $k_0\in J_3$. Then the following (a) and (b) are equivalent:

(a) K satisfies the condition (RI):

(RI) $k_0\notin K_\alpha$ for each $\alpha\in R(A)-C(B)$.

(b) L satisfies the condition (CNSF):

(CNSF) $L_{(2,3)}(A\times\{k_0\})=B$.

By the same way as Proposition 5, we can easily show the following Proposition 70.

Proposition 70.

Let $f:J_1\times J_2\to J_3$ be a Latin square map. Let $K=\{K_\alpha\}_{\alpha\in J_1\times J_2}\in \boldsymbol{LMTX}(f)$. Let $S\subset \boldsymbol{b}\subset J_1\times J_2$ be a set with

(NSF) $|K(S)|=|S|$.

If $\boldsymbol{b}$ is a row or a column, then $K^*=\{K^*_\alpha\}_{\alpha\in J_1\times J_2}\in \boldsymbol{LMTX}(f)$. Here,

$$K^*_\alpha=\begin{cases}K_\alpha-K(S) & for\ \alpha\in \boldsymbol{b}-S\\ K_\alpha & for\ \alpha\in J_1\times J_2-(\boldsymbol{b}-S)\end{cases}.$$

Proposition 71.

Let $f:J_1\times J_2\to J_3$ be a Latin square map and $g=\omega_{(1,3)}(f)$. Let $A\subset J_1$, $B\subset J_2$ $|A|=|B|=n$ and $k_0\in J_3$. Let $K=\{K_\alpha\}_{\alpha\in J_1\times J_2}\in \boldsymbol{LMTX}(f)$ and $L_{(1,3)}=\boldsymbol{MX}_{(1,3)}(f)(K)\in \boldsymbol{LMTX}(g)$, $L_{(1,3)}=\{L_\beta\}_{\beta\in J_3\times J_2}$. If K satisfies the condition (CI), then we have the followings:

(a) $K^*=(K^*_\alpha)_{\alpha\in J_1\times J_2}\in \boldsymbol{LTMX}(f)$ with

$$K^*_\alpha=\begin{cases}K_\alpha-\{k_0\} & for\ \alpha\in R(A)-C(B)\\ K_\alpha & for\ \alpha\in J_1\times J_2-(R(A)-C(B))\end{cases}$$

(b) $L^*_{(1,3)}=(L^*_\beta)_{\beta\in J_3\times J_2}\in \boldsymbol{LTMX}(g)$ with

$$L^*_\beta=\begin{cases}L_\beta-A & for\ \ \beta\in\{k_0\}\times(J_2-B)\\ L_\beta & for\ \ \beta\in J_3\times J_2-(\{k_0\}\times(J_2-B))\end{cases}$$

(c) $K^* = MX_{(1,3)}(g)(L^*_{(1,3)})$, $L^*_{(1,3)} = MX_{(1,3)}(f)(K^*)$.

Proof.

By Proposition 67, we have (RC). Hence we have (a).

By Proposition 68, $L_{(1,3)}$ satisfies (RNSF). Thus, by Proposition 70, we have (b).

Proof of (c).

Now, we continue our arguments by using same notations in the proof of Proposition 68.

By the definition, we have that

(26) $L^*_{(1,3)} \leqq L_{(1,3)}$.

By (b) we can put

$M_{(1,3)} = MX_{(1,3)}(g)(L_{(1,3)}) \in \boldsymbol{LTMX}(\omega_{(1,3)}(g))$,

$M^*_{(1,3)} = MX_{(1,3)}(g)(L^*_{(1,3)}) \in \boldsymbol{LTMX}(\omega_{(1,3)}(g))$ and

$M^*_{(1,3)} = (M^*_\alpha)_{\alpha \in J_1 \times J_2}$.

By (26) and Proposition 66, we have that

(27) $M^*_{(1,3)} \leqq M_{(1,3)}$.

By (25), (27) we have that

(28) $M^*_{(1,3)} \leqq K$.

By definitions, we have that

(29) $M^*_\alpha = \{k \in J_3 : i \in L^*_{(k,j)}\}$

for each $\alpha = (i,j) \in J_1 \times J_2$.

Since $L^*_\beta \subset L_\beta$ for each $\beta \in J_3 \times J_2$ by (26), by (19), (29) we have that

Claim 5.

$M^*_\alpha \subset M_\alpha$ for each $\alpha \in J_1 \times J_2$.

Next, we show that

Claim 6.

$k_0 \notin M^*_\alpha$ for each $\alpha \in R(A) - C(B)$.

We assume that Claim 6 does not hold. That is,

(30) $k_0 \in M^*_{\alpha_u}$ for some $\alpha_u \in R(A) - C(B)$.

Put $\alpha_u = (i_u, j_u) \in R(A) - C(B)$. Then we have

(31) $i_u \in A$ and $j_u \notin B$.

By (29) we have

(32) $M^*_{\alpha_u} = \{k \in J_3 : i_u \in L^*_{(k,j_u)}\}$.

By (30), (32) we have that

(33) $i_u \in L^*_{(k_0, j_u)}$.

By (31) we have that

(34) $(k_0, j_u) \in \{k_0\} \times J_2 - \{k_0\} \times B = \{k_0\} \times J_2 - S$.

By (b), (33), (34) we have that

(35) $i_u \in L^*_{(k_0, j_u)} = L_{(k_0, j_u)} - A$.

Thus, by (35) we have that

(36) $i_u \notin A$.

(36) contradicts to (31). Hence, we have Claim 6.

Claim 7.

$M_{\alpha_u} - \{k_0\} \subset M^*_{\alpha_u}$ for each $\alpha_u \in R(A) - C(B)$.

We assume that Claim 7 does not hold. Then we have

(37) $M_{\alpha_u} - \{k_0\} \not\subset M^*_{\alpha_u}$

for some $\alpha_u = (i_u, j_u) \in R(A) - C(B)$.

By (37) we have that

(38) $i_u \in A$ and $j_u \notin B$.

By (37) there exists a $k_u \in J_3$ with

(39) $k_u \in M_{\alpha_u} - \{k_0\}$ and $k_u \notin M^*_{\alpha_u}$.

Since $k_u \in M_{\alpha_u} - \{k_0\}$ by (39), by (19) we have

(40) $i_u \in L_{(k_u, j_u)}$ and $k_u \neq k_0$.

Since $k_u \notin M^*_{\alpha_u}$ by (41), by (31) we have that

(41) $i_u \notin L^*_{(k_u, j_u)}$.

Since $k_u \neq k_0$ by (40), we have that

(42) $(k_u, j_u) \notin \{k_0\} \times J_2 - S$.

By (b), (42) we have that

(43) $L^*_{(k_u, j_u)} = L_{(k_u, j_u)}$.

By (41), (43) we have that

(44) $i_u \notin L^*_{(k_u, j_u)} = L_{(k_u, j_u)}$.

(44) contradicts to (40). Hence, we have Claim 7.

Claim 8.

$M^*_{\alpha_u} = M_{\alpha_u} - \{k_0\}$ for each $\alpha_u \in R(A) - C(B)$.

Take any $\alpha_u \in R(A) - C(B)$, by Claim 5 we have taht

(45) $M^*_{\alpha_u} \subset M_{\alpha_u}$.

By (45) we have that

(46) $M^*_{\alpha_u} - \{k_0\} \subset M_{\alpha_u} - \{k_0\}$.

By Claim 6 we have that

(47) $M^*_{\alpha_u} - \{k_0\} = M^*_{\alpha_u}$.

By (46), (47) we have that

(48) $M^*_{\alpha_u} \subset M_{\alpha_u} - \{k_0\}$.

By Claim 7 and (48), we hae that

(49) $M^*_{\alpha_u} = M_{\alpha_u} - \{k_0\}$.

By (49) we have Claim 8.

Claim 9.
$M^*_\alpha \supset M_\alpha$ for each $\alpha \in J_1 \times J_2 - (R(A) - C(B))$.
We assume that Claim 9 does not hold. Then there is an $\alpha_v = (i_v, j_v) \in J_1 \times J_2 - (R(A) - C(B))$ such that
(50) $M^*_{\alpha_v} \not\supset M_{\alpha_v}$.
By (50) there exists a $k_v \in J_3$ such that
(51) $k_v \in M_{\alpha_v}$ and
(52) $k_v \notin M^*_{\alpha_v}$.
By (19), (51) we have that
(53) $i_v \in L_{(k_v, j_v)}$.
By (32), (52) we have that
(54) $i_v \notin L^*_{(k_v, j_v)}$.
Since $\alpha_v = (i_v, j_v) \notin R(A) - C(B)$, we have that
(55) $i_v \notin A$ or
(56) $i_v \in A$ and $j_v \in B$.

We have the two cases as follows:
(57) $k_v = k_0$, or
(58) $k_v \neq k_0$.

Thus we consider the following cases:
Case 1. (57) and (55) are hold.
Case 2. (57) and (56) are hold.
Case 3. (58) and (55) are hold.
Case 4. (58) and (56) are hold.

We consider Case 1.
Thus, we have that
(59) $k_v = k_0$ and $i_v \notin A$.
Now (59) divided into the cases:
(60) $k_v = k_0$, $i_v \notin A$ and $j_v \in B$.
(61) $k_v = k_0$, $i_v \notin A$ and $j_v \notin B$.

The case (60).
By (60), we have that
(62) $(k_v, j_v) = (k_0, j_v) \in \{k_0\} \times B = S$.
By (b), (62) we have that
(63) $L^*_{(k_v, j_v)} = L_{(k_v, j_v)}$.
By (53), (54), (63) we have that
(64) $i_v \in L_{(k_v, j_v)} = L^*_{(k_v, j_v)} \not\ni i_v$.
(64) has a contradiction. Hence, (60) does not happen.

The case (61).
By (61) we have that
(65) $(k_v, j_v) = (k_0, j_v) \in \{k_0\} \times J_2 - S$.
By (b), (65) we have that
(66) $L^*_{(k_v, j_v)} = L_{(k_v, j_v)} - A$.
By (54), (66) we have that
(67) $i_v \notin L^*_{(k_v, j_v)} = L_{(k_v, j_v)} - A$.
By (53), (67) we have that
(68) $i_v \in A$.
(68) contradicts to (61).
Hence, (61) does not happen. Therefore, Case 1 does not happen.

The case 2.
We have that
(69) $k_v = k_0$, $i_v \in A$ and $j_v \in B$.
By (69) we have that
(70) $(k_v, j_v) = (k_0, j_v) \in \{k_0\} \times B = S$.
By (b), (70) we have that
(71) $L^*_{(k_v, j_v)} = L_{(k_v, j_v)}$.
By (53), (54), (71) we have that
(72) $i_v \in L_{(k_v, j_v)} = L^*_{(k_v, j_v)} \not\ni i_v$.
(72) has a contradiction. Hence, case 2 does not happen.

The case 3.
We have that
(73) $k_v \neq k_0$, $i_v \notin A$.
By (73) we have that
(74) $(k_v, j_v) \in J_3 \times J_2 - (\{k_0\} \times J_2 - S)$.
By (b) ,(74) we have that
(75) $L^*_{(k_v, j_v)} = L_{(k_v, j_v)}$.
By (53), (54), (75) we have that
(76) $i_v \in L_{(k_v, j_v)} = L^*_{(k_v, j_v)} \not\ni i_v$.
(76) has a contradiction. Hence, the case 3 does not happen.

The case 4.
We have that
(77) $k_v \neq k_0$, $i_v \in A$ and $j_v \in B$.
By (77) we have that
(78) $(k_v, j_v) \in J_3 \times J_2 - (\{k_0\} \times J_2 - S)$.
By (b), (78) we have that
(79) $L^*_{(k_v, j_v)} = L_{(k_v, j_v)}$.
By (53), (54), (79) we have that
(80) $i_v \in L_{(k_v, j_v)} = L^*_{(k_v, j_v)} \not\ni i_v$.
(80) has a contradiction. Hence, case 4 does not happen.
Hence, all cases 1−4 do not happen. Therefore we have Claim 9.

By Claim 5 and Claim 9 we have that

Claim 10.
$M^*_\alpha = M_\alpha$ for each $\alpha \in J_1 \times J_2 - (R(A) - C(B))$.

Since $L_{(13)} = MX_{(1,3)}(f)(K)$ and $M_{(1,3)} = MX_{(1,3)}(g)(L_{(1,3)})$, by Proposition 66, we have that

Claim 11.
$K = M_{(1,3)}$, i.e., $K_\alpha = M_\alpha$ for each $\alpha \in J_1 \times J_2$.

By Claim 8 , Claim 10 and Claim 11 we have

Claim 12.
$K^* = M^*_{(1,3)}$ i.e., $K^* = MX_{(1,3)}(g)(L^*_{(1,3)})$.

By Claim 12, Proposition 66 we have

$$MX_{(1,3)}(f)(K^*) = MX_{(1,3)}(f) \circ MX_{(1,3)}(g)(L^*_{(1,3)}) = 1(L^*_{(1,3)}) = L^*_{(1,3)}.$$

Thus we have (c).
Hence, we have Proposition 71.

Proposition 72.
Let $f: J_1 \times J_2 \to J_3$ be a Latin square map and $h = \omega_{(2,3)}(f)$. Let $A \subset J_1$, $B \subset J_2$ $|A| = |B| = n$ and $k_0 \in J_3$. Let $K = \{K_\alpha\}_{\alpha \in J_1 \times J_2} \in LMTX(f)$ and $L_{(2,3)} = MX_{(2,3)}(f)(K) \in LMTX(h)$, $L_{(2,3)} = \{L_\beta\}_{\beta \in J_1 \times J_3}$. If K satisfies the condition (RI), then we have the followings:

(a) $K^* = (K^*_\alpha)_{\alpha \in J_1 \times J_2} \in LTMX(f)$ with

$$K^*_\alpha = \begin{cases} K_\alpha - \{k_0\} & for\ \alpha \in C(B) - R(A) \\ K_\alpha & for\ \alpha \in J_1 \times J_2 - (C(B) - R(A)) \end{cases}$$

(b) $L^*_{(2,3)} = (L^*_\beta)_{\beta \in J_1 \times J_3} \in LTMX(h)$ with

$$L^*_\beta = \begin{cases} L_\beta - B & for\ \beta \in (J_1 - A) \times \{k_0\} \\ L_\beta & for\ \beta \in J_1 \times J_3 - ((J_1 - A) \times \{k_0\}) \end{cases}$$

(c) $K^* = MX_{(2,3)}(h)(L^*_{(2,3)})$, $L^*_{(2,3)} = MX_{(2,3)}(f)(K^*)$.

By the similar ways as Proposition 71 we can show Proposition 72.

Proposition 73.
Proposition 70 implies Proposition 67.
Proof. First, we consider (a) in Proposition 67. Thus $K \in LMTX(f)$ has the condition (CI). By Proposition 68, $L_{(1,3)} = MX_{(1,3)}(f)(K)$ satisfies the condition (RNSF).
By Proposition 70, $L^*_{(1,3)} \in LMTX(g)$, $g = \omega_{(1,3)}(f)$.
Hence, $L^*_{(13)}$ satisfies (b) of Proposition 71.
By the proof of (c) in Proposition 71, we showed Claim 12 in the proof of Proposition 71. That is, $M^*_{(1,3)} = MX_{(1,3)}(g)(L^*_{(1,3)}) \in LMTX(f)$ and $M^*_{(1,3)} = K^*$.
Hence, $K^* \in LMTX(f)$, i.e., we have (RC). Thus, we have (a).
By similar way we can show (b) in Proposition 67.
Hence, we have Proposition 73.

Proposition 74.
(a) The following conditions are equivalent:
(CI) $k_0 \not\in K_\alpha$ for each $\alpha \in C(B) - R(A)$.
(RI) $k_0 \not\in K_\alpha$ for each $\alpha \in R(J_1 - A) - C(J_2 - B)$.
(b) The following conditions are equivalent:
(RI) $k_0 \not\in K_\alpha$ for each $\alpha \in R(A) - C(B)$.
(CI) $k_0 \not\in K_\alpha$ for each $\alpha \in C(J_2 - B) - R(J_1 - A)$.
Proof. We show (a).
We have that
(1) $C(B) - R(A) = J_1 \times B - A \times J_2$
$= J_1 \times B - A \times J_2 \cap J_1 \times B$
$= J_1 \times B - A \times B = (J_1 - A) \times B$.
(2) $R(J_1 - A) - C(J_2 - B)$
$= (J_1 - A) \times J_2 - J_1 \times (J_2 - B)$
$= (J_1 - A) \times J_2 - (J_1 \times (J_2 - B)) \cap ((J_1 - A) \times J_2)$
$= (J_1 - A) \times J_2 - (J_1 - A) \times (J_2 - B)$
$= (J_1 - A) \times (J_2 - (J_2 - B)) = (J_1 - A) \times B$.
By (1), (2) we have
(3) $C(B) - R(A) = R(J_1 - A) - C(J_2 - B)$.
By (3) we have (a).
By the same way we show (b).
Hence, we have Proposition 74.

Remark 75.
(1) By Proposition 73 we can say that *n − koku − domei Theorem* (Proposition 70) implies *n − igeta Theorem* (Proposition 67).
(2) When we observe *n − igeta Theorem*, by Proposition 73, we investigate only *n − igeta Theorem* for $n = 1,2,3,4$.

References
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